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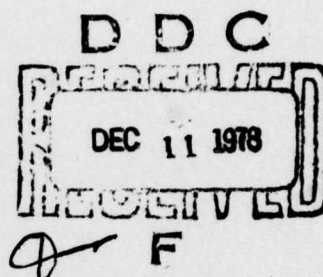
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Interim Scientific Report



FINITE ELEMENTS FOR FLUID DYNAMICS  
Mixed-type problems, Transonic flows.

Nima Geffen  
Principal Investigator

Department of Applied Mathematics  
Tel-Aviv University  
Ramat-Aviv, Israel

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The report includes two parts: I. Various finite elements formulations and discretizations for initial/boundary value problems for the Tricomi equation in the hyperbolic domain, a continuation of our work last year. II. An attempt to extend "hyperbolic" methods of analysis to mixed type flows with hyperbolic-elliptic shocks and ensuing entropy conditions for stable, accurate numerical schemes, by Professor Michael Mock with an appendix by Lustman and Geffen on the impossibility of elliptic-elliptic shocks (a fact well-known to any aerodynamicist and rather obvious and physical grounds.) The question of entropy		



functions and inequalities is inherently related to the well posedness of the non-dissipative formulation and in turn, we feel, to the character and behaviour of the different variational functionals.

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## FORWARD

Aiming at understanding and simulating mixed-type flows of interest in applications to transonic aerodynamics and plasmadynamics, mathematical and numerical analyses for relevant initial boundary value problems and finite element discretizations are described in the following.

The report is divided into two parts:

I. Various finite elements formulations and discretizations for initial/boundary value problems for the Tricomi equation in the hyperbolic domain, work carried out by Mrs. Frieda Loinger. This is a continuation to Sara Yaniv's work reported last year, and the schemes developed are tried on the same sample problems.

II. An attempt to extend "hyperbolic" methods of analysis to mixed type flows with hyperbolic-elliptic shocks and ensuing entropy conditions for stable, accurate numerical schemes, done by Prof. Michael Mock with an appendix by Lustman and Geffen on the impossibility of elliptic-elliptic shocks (a fact well-known to any aerodynamicist and rather obvious on physical grounds.)

The question of entropy functions and inequalities is inherently related to the well posedness of the nondissipative formulation and in turn, we feel, to the character and behaviour of the different variational functionals. A further clarification along this way may be essential for the ability to correctly select the physically relevant fields from the multitude of mathematically possible ones (e.g. with expansion shocks) by full-proof a-priori conditions in the simulation scheme.



- 2 -

FE DISCRETIZATION FOR  
TRICOMI PROBLEMS IN A HYPERBOLIC REGION

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## INTRODUCTION

A finite-element formulation of Tricomi's problem using bilinear isoparametric characteristic rectangular elements in the hyperbolic domain is given in [1] with two sample problems. In this report discretizations for linear triangular elements are given, with calculations for the hyperbolic part of the sample problems mentioned above. The following discretizations are given in detail:

I. Global application of the variational conditions for  
a) Cauchy b) Goursat conditions resulting in an implicit scheme for the whole hyperbolic triangle.

II. Local application of the variational conditions on each element for a) Cauchy b) Goursat boundaries resulting in an implicit scheme (or marching along characteristics).

The order of the approximations is:

$O(h^2)$  at internal points for all implicit schemes,  $O(h)$  near external boundaries and  $O(h^{2/3})$  near the parabolic line  $y=0$ . For the explicit (marching) schemes, only  $O(h)$  is achieved. The isoparametric schemes become inconsistent near the parabolic line, due to the singular behavior (coalescence) of the characteristics there; hence the isoparametric triangles are changed to regular  $(x,y)$  triangles for which consistency is restored. It also facilitates further local refinements, (possibly using the multi-grid method), where they may be needed.

An appropriate combination of linear triangles near the para-

bolic line and isoparametric characteristic triangles away from it give the most accurate results, with the marching scheme being most efficient and easiest to apply. Schemes using Goursat's conditions have been tried. Convergence has been demonstrated for implicit schemes [1], but not for explicit ones. Details will be reported.

#### Reference

[1] Yaniv, S., Variational Formulation for Formally Symmetric Problems and its Application to the Tricomi Equation, Final Scientific Report, Grant AFOSR 73-256, July 1977, Part I, also Ph.D. Thesis, Tel-Aviv University, Israel, 1978.

1. The hyperbolic problem

Given Tricomi's equation:

$$y\varphi_{xx} - \varphi_{yy} = 0 \quad (1)$$

or, equivalently, the first order system

$$yu_x - v_y = 0 \quad (1a)$$

$$u_y - v_x = 0 \quad (1b)$$

where:

$$(u,v) = (\varphi_x, \varphi_y),$$

we look for a solution in the hyperbolic region  $D$ , bounded by the parabolic line  $y = 0$  and the two characteristics  $\Gamma_1$  and  $\Gamma_2$  where (figure 1):

$$\Gamma_1: -1 = \eta = x - \frac{2}{3}y^{3/2} \quad -1 \leq x \leq 0$$

and

$$\Gamma_2: +1 = \xi = x + \frac{2}{3}y^{3/2} \quad 0 \leq x \leq 1.$$

We assume that the problem is well-posed, with either Cauchy (i) or Goursat (ii) conditions given:

$$(i) \quad \varphi(x,0) = f_1(x) \quad \varphi_y(x,0) = f_2(x)$$

or equivalently  $u(x,0) = f_1'(x)$   $v(x,0) = f_2(x)$   
given on the parabolic line.

(ii)  $\varphi(x, y(x)) = f(x)$ , or equivalently  $u\sqrt{y}+v = g(y)$  given on the  
characteristic  $\Gamma_1$  ( $y = [\frac{3}{2}(x+1)]^{2/3}$ )  
and:  $\varphi(x,0) = f_1(x)$  or equivalently  $u(x,0) = f_1'(x)$  on the  
parabolic line.

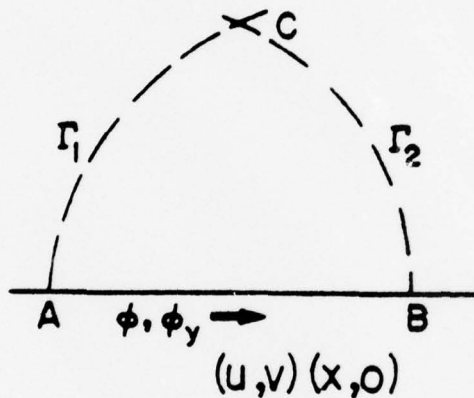


Figure 1a : Cauchy data.

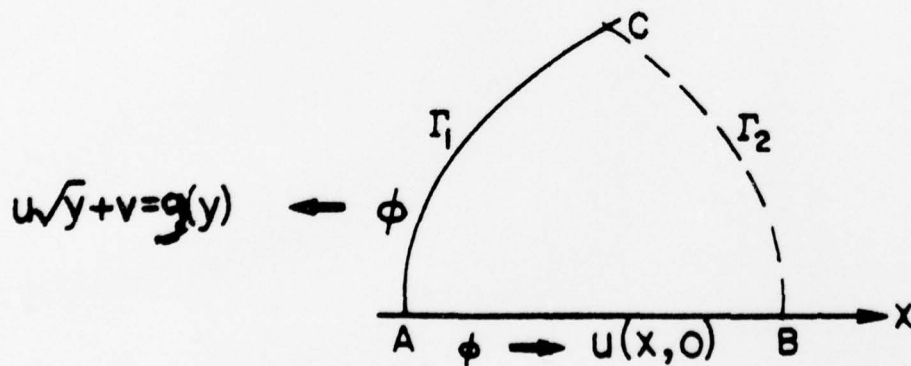


Figure 1b : Goursat prob.



## 2. Variational formulation in the hyperbolic domain

Define the functional

$$(2) \quad J(u,v,\lambda) = \iint_D [yu^2 - v^2 + \lambda(u_y - v_x)] dx dy + \int_{\Gamma_m} \lambda(udx + vdy)$$

where  $D$  is the hyperbolic domain and  $\Gamma_m$  includes all surfaces where no boundary conditions are given:  $\Gamma_m = \Gamma_1 \cup \Gamma_2$  for the Cauchy problem and  $\Gamma_m = \Gamma_2$  for the Goursat problem;  $\lambda$  is kept fixed on  $\Gamma_m$ .

$J(u,v,\lambda)$  becomes stationary when  $(u,v,\lambda)$  satisfy the equations:

$$\begin{aligned} (2a) \quad & 2yu = \lambda_y \\ (2b) \quad & 2v = \lambda_x \\ (2c) \quad & u_y - v_x = 0 \end{aligned} \quad \left. \begin{array}{l} 2yu = \lambda_y \\ 2v = \lambda_x \end{array} \right\} \longleftrightarrow yu_x = v_y$$

which are equivalent to equations (1) for  $\lambda(x,y) \in C^2$ .

### 3. Finite Element Schemes

The hyperbolic region is divided into triangular isoparametric elements, (figure 1). In every element we take trial functions which are linear in  $\xi$  and  $\eta$ .

$$\psi_m^e = p_m^e \eta + q_m^e \xi + r_m^e$$

where

$$\psi^e = (u, v, \lambda)$$

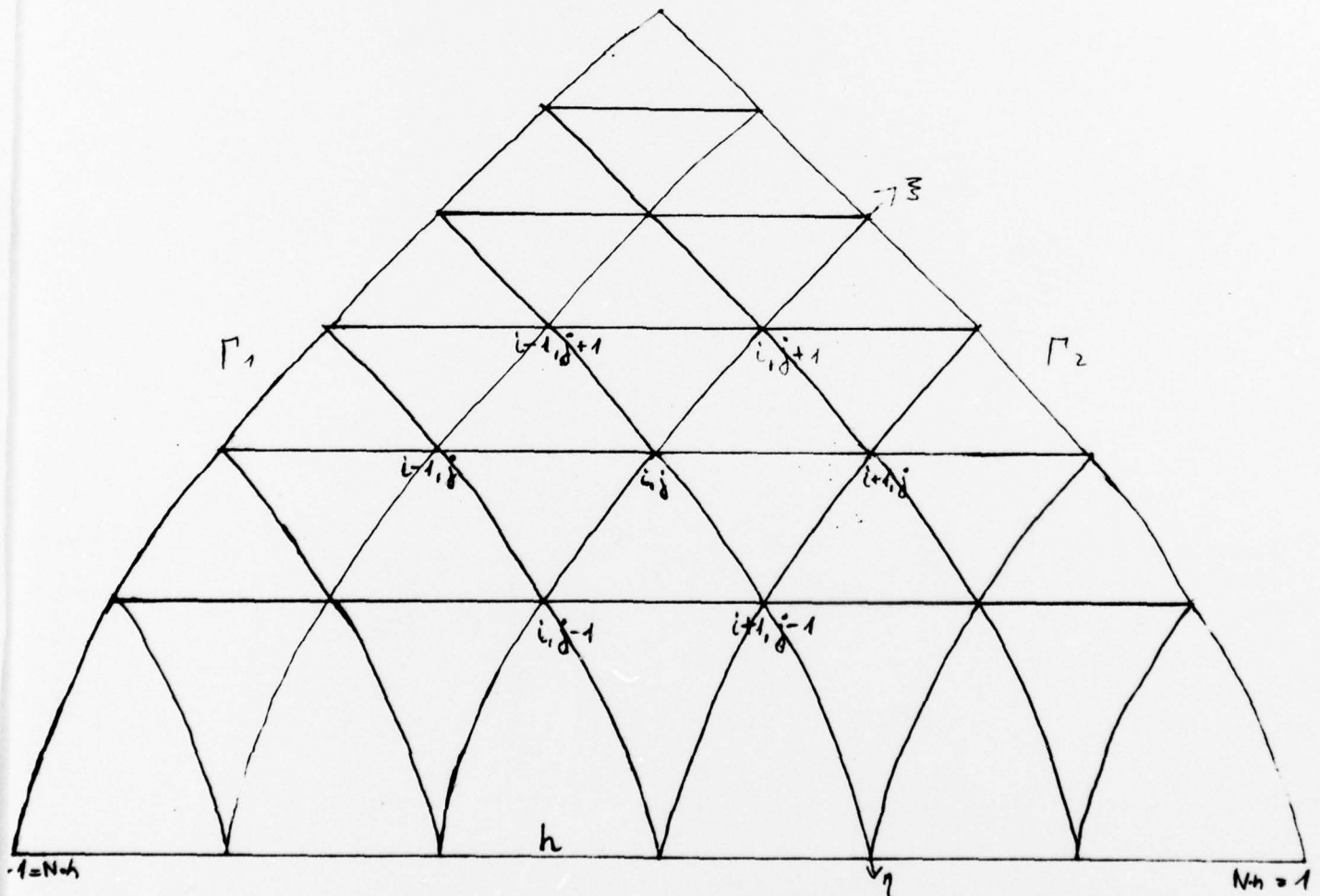


Figure 2: Triangular Isoparametric Elements

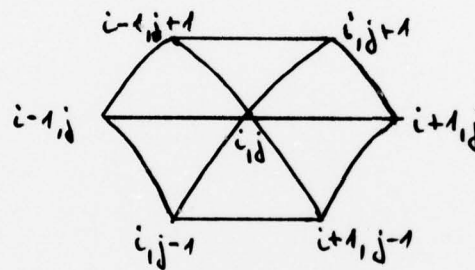
where  $y_{i,j} = y_j = (\frac{3}{4}hj)^{2/3}$ ,  $x_{i,j} = (i+\frac{j}{2})h$   
and  $\eta_{i,j} = ih$ ,  $\xi_{i,j} = (i+j)h$

A. Interior points

Minimizing the functional for interior points we get the following schemes:

$$j = 1, \dots, 2N-2$$

$$i = -N+1, \dots, N-1-j$$



$$\frac{\partial}{\partial u_{i,j}} J(u, v, \lambda) = 0 \Rightarrow$$

$$\begin{aligned} & \left(\frac{3}{4}h\right)^{4/3} \cdot \frac{12}{7} [(u_{i,j+1} + u_{i-1,j+1}) \left(\frac{1}{2}j^{7/3} + \frac{1}{2}(j+1)^{7/3} + \frac{3}{10}j^{10/3} - \frac{3}{10}(j+1)^{10/3}\right) \\ & + (u_{i-1,j} + u_{i+1,j}) \left(-j^{7/3} - \frac{9}{65}j^{13/3} + \frac{9}{130}(j-1)^{13/3} + \frac{9}{130}(j+1)^{13/3}\right) \\ & + u_{i,j} \left(-2j^{7/3} + \frac{3}{5}(j+1)^{10/3} - \frac{3}{5}(j-1)^{10/3} \right. \\ & \left. + \frac{18}{65}j^{13/3} - \frac{9}{65}(j-1)^{13/3} - \frac{9}{65}(j+1)^{13/3}\right) \end{aligned} \quad (3a)$$

$$\begin{aligned}
 & + (u_{i,j-1} + u_{i+1,j-1}) \left( \frac{1}{2}j^{7/3} + \frac{1}{2}(j-1)^{7/3} + \frac{3}{10}(j-1)^{10/3} - \frac{3}{10}j^{10/3} \right) ] \\
 & = (\lambda_{i-1,j+1} - \lambda_{i,j-1}) + (\lambda_{i,j+1} - \lambda_{i+1,j-1})
 \end{aligned}$$

The scheme (3a) approximates the equation  $2yu = \lambda_y$  at the mesh point  $(x_{i,j}, y_{i,j})$ , and the accuracy is:

$$2yu - \lambda_y = O(h^2)$$

$$\frac{\partial}{\partial v_{i,j}} J(u, v, \lambda) = 0 \rightarrow$$

$$\begin{aligned}
 & (v_{i,j+1} + v_{i-1,j+1}) \left( j^{5/3} + (j+1)^{5/3} + \frac{3}{4}j^{8/3} - \frac{3}{4}(j+1)^{8/3} \right) \\
 & + (v_{i-1,j} + v_{i+1,j}) \left( -2j^{5/3} - \frac{9}{22}j^{11/3} + \frac{9}{44}(j-1)^{11/3} + \frac{9}{44}(j+1)^{11/3} \right) \\
 & + v_{i,j} \left( -4j^{5/3} + \frac{3}{2}(j+1)^{8/3} - \frac{3}{2}(j-1)^{8/3} - \frac{9}{22}(j-1)^{11/3} - \frac{9}{22}(j+1)^{11/3} + \frac{9}{11}j^{11/3} \right) \\
 & + (v_{i,j-1} + v_{i+1,j-1}) \left( (j-1)^{5/3} + j^{5/3} + \frac{3}{4}(j-1)^{8/3} - \frac{3}{4}j^{8/3} \right) \\
 & = \frac{1}{h} [ (\lambda_{i,j+1} - \lambda_{i-1,j+1}) \left( j^{5/3} + (j+1)^{5/3} + \frac{3}{4}j^{8/3} - \frac{3}{4}(j+1)^{8/3} \right) \\
 & + (\lambda_{i+1,j} - \lambda_{i-1,j}) \left( -2j^{5/3} + \frac{3}{8}(j+1)^{8/3} - \frac{3}{8}(j-1)^{8/3} \right) \\
 & + (\lambda_{i+1,j-1} - \lambda_{i,j-1}) \left( (j-1)^{5/3} + j^{5/3} + \frac{3}{4}(j-1)^{8/3} - \frac{3}{4}j^{8/3} \right) ]
 \end{aligned} \tag{3b}$$

The scheme (3b) approximates the equation  $2v = \lambda x$  at the mesh point  $(x_{i,j}, y_{i,j})$  and the accuracy is:

$$2v - \lambda_x = O(h^2)$$

$$\frac{\partial}{\partial \lambda_{i,j}} J(u, v, \lambda) = 0 \Rightarrow$$

$$\begin{aligned} & \frac{12}{5} \cdot \frac{(\frac{3}{4})^{2/3}}{h^{1/3}} [(v_{i,j+1} - v_{i-1,j+1})(j^{5/3} + (j+1)^{5/3} + \frac{3}{4}j^{8/3} - \frac{3}{4}(j+1)^{8/3}) \\ & + (v_{i+1,j} - v_{i-1,j})(-2j^{5/3} + \frac{3}{8}(j+1)^{8/3} - \frac{3}{8}(j-1)^{8/3}) \\ & + (v_{i+1,j-1} - v_{i,j-1})(j^{5/3} + (j-1)^{5/3} + \frac{3}{4}(j-1)^{8/3} - \frac{3}{4}j^{8/3})] \\ & = (u_{i-1,j+1} - u_{i,j-1}) + (u_{i,j+1} - u_{i+1,j-1}). \end{aligned} \quad (3c)$$

The scheme (3c) approximates the equation  $u_y = v_x$  at the mesh point  $(x_{i,j}, y_{i,j})$  and the accuracy is;

$$u_y - v_x = O(h^2)$$

Remark:

For an interior point  $(x, y)$  the accuracy of the equations (2a)-(2c) is  $O(h^2)$  only if we assume that the value of  $j$  is sufficiently large. This can always be achieved by taking smaller values of  $h$ , because  $y = (\frac{3}{4}hj)^{2/3}$  so  $j$  increases if  $h$  decreases for a fixed interior point.



B. Boundary Points for the Cauchy problem

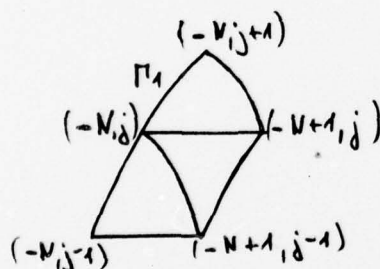
In this case  $\Gamma_m = \Gamma_1 \cup \Gamma_2$  and we have the following functional

$$J_1(u, v, \lambda) = \iint_D [yu^2 - v^2 + \lambda(u_y - v_x)] dx dy + \int_{\Gamma_1 \cup \Gamma_2} \lambda(udx + vdy)$$

mesh points on  $\Gamma_1$  :  $i = -N$

$$j = 1, \dots, 2N-1$$

$$\eta = -1$$



$$\frac{\partial}{\partial u_{-N, j}} J_1(u, v, \lambda) = 0 \Rightarrow$$

$$\left(\frac{3}{4}h\right)^{4/3} \frac{18}{7} [u_{-N, j+1} \left(\frac{1}{2}j^{7/3} + \frac{3}{5}j^{10/3} + \frac{3}{10}(j+1)^{10/3} + \frac{27}{130}j^{13/3} - \frac{27}{130}(j+1)^{13/3}\right)$$

$$+ u_{-N+1, j} \left(-j^{7/3} - \frac{9}{65}j^{13/3} + \frac{9}{130}(j-1)^{13/3} + \frac{9}{130}(j+1)^{13/3}\right)$$

$$+ u_{-N, j} \left(-j^{7/3} - \frac{3}{5}(j-1)^{10/3} - \frac{6}{5}j^{10/3} + \frac{9}{65}j^{13/3} - \frac{18}{65}(j-1)^{13/3} + \frac{9}{65}(j+1)^{13/3}\right)$$

$$\begin{aligned}
 & + u_{-N+1,j-1} \left( \frac{1}{2}j^{7/3} + \frac{1}{2}(j-1)^{7/3} + \frac{3}{10}(j-1)^{10/3} - \frac{3}{10}j^{10/3} \right) \\
 & + u_{-N,j-1} \left( \frac{1}{2}(j-1)^{7/3} + \frac{3}{5}(j-1)^{10/3} + \frac{3}{10}j^{10/3} + \frac{27}{130}(j-1)^{13/3} - \frac{27}{130}j^{13/3} \right) ] \\
 & = (\lambda_{-N,j+1} - \lambda_{-N+1,j-1}) + (\lambda_{-N,j} - \frac{\lambda_{-N,j-1} + \lambda_{-N+1,j-1}}{2}).
 \end{aligned}$$

The accuracy is:

$$2yu - \lambda_y = 0(h)$$

$$\frac{\partial}{\partial v_{-N,j}} J_1(u, v, \lambda) = 0 \Rightarrow$$

$$\begin{aligned}
 & v_{-N,j+1} \left( j^{5/3} + \frac{3}{2}j^{8/3} + \frac{3}{4}(j+1)^{8/3} + \frac{27}{44}j^{11/3} - \frac{27}{44}(j+1)^{11/3} \right) \\
 & + v_{-N+1,j} \left( -2j^{5/3} - \frac{9}{22}j^{11/3} + \frac{9}{44}(j-1)^{11/3} + \frac{9}{44}(j+1)^{11/3} \right) \\
 & + v_{-N,j} \left( -2j^{5/3} - \frac{3}{2}(j-1)^{8/3} - 3j^{8/3} - \frac{9}{11}(j-1)^{11/3} + \frac{9}{22}j^{11/3} + \frac{9}{22}(j+1)^{11/3} \right) \\
 & + v_{-N+1,j-1} \left( (j-1)^{5/3} + j^{5/3} + \frac{3}{4}(j-1)^{8/3} - \frac{3}{4}j^{8/3} \right) \\
 & + v_{-N,j-1} \left( (j-1)^{5/3} + \frac{3}{2}(j-1)^{8/3} + \frac{3}{4}j^{8/3} + \frac{27}{44}(j-1)^{11/3} - \frac{27}{44}j^{11/3} \right) \\
 & = \frac{1}{h} [(\lambda_{-N+1,j} - \lambda_{-N,j}) \left( -2j^{5/3} + \frac{3}{8}(j+1)^{8/3} - \frac{3}{8}(j-1)^{8/3} \right) \\
 & + (\lambda_{-N+1,j-1} - \lambda_{-N,j-1}) \left( j^{5/3} + (j-1)^{5/3} + \frac{3}{4}(j-1)^{8/3} - \frac{3}{4}j^{8/3} \right)
 \end{aligned}$$

Accuracy:

$$2v - \lambda_x = 0(h)$$

$$\frac{\partial}{\partial \lambda_{-N,j}} J_1(u, v, \lambda) = 0 \Rightarrow$$

$$(u_{-N,j+1} - u_{-N+1,j-1}) + (u_{-N,j} \frac{u_{-N,j-1} + u_{-N+1,j-1}}{2})$$

$$= \frac{18}{5} \cdot \frac{(\frac{3}{4})^{2/3}}{h^{1/3}} [(v_{-N+1,j} - v_{-N,j}) (-2j^{5/3} + \frac{3}{8}(j+1)^{8/3} - \frac{3}{8}(j-1)^{8/3})$$

$$+ (v_{-N+1,j-1} - v_{-N,j-1}) (j^{5/3} + (j-1)^{5/3} + \frac{3}{4}(j-1)^{8/3} - \frac{3}{4}j^{8/3})]$$

Accuracy:

$$u_y - v_x = 0(h)$$

And similarly for  $\Gamma_2$ .

#### 4. Linear Triangular Elements near the Parabolic Line

To avoid the inconsistency near the parabolic line for small values of  $j$ , (near  $j = 0$ ), we must change the trial functions near the parabolic line for the same mesh points

$$x_{i,j} = (i + \frac{j}{2})h$$

$$y_{i,j} = y_j = (\frac{3}{4}hj)^{2/3}$$

and take ordinary triangular elements with linear variation in  $x$  and  $y$ , and linear approximations to  $\Gamma_1$  and  $\Gamma_2$  near the parabolic line.

Let the approximated domain be  $\bar{D}$ , and the approximated characteristics,  $\bar{\Gamma}_1$  and  $\bar{\Gamma}_2$ .

For  $j \leq j_0$  the trial functions will be linear in  $x, y$ .  
For  $j \geq j_0$  the trial functions will be linear in  $\xi, \eta$ .

The variational formulation is:

$$\bar{J}(u, v, \lambda) = \iint_{\bar{D}} [yu^2 - v^2 + \lambda(u_y - v_x)] dx dy + \int_{\bar{\Gamma}} \lambda(udx + vdt)$$

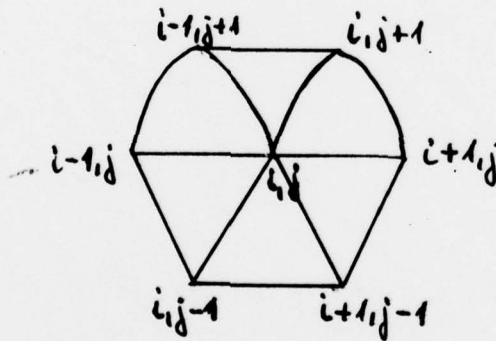
For  $j \geq j_0 + 1$  all the elements belonging to the mesh point  $(i, j)$  are isoparametric, so the schemes are the same as in chapter 3.

For  $\underline{j = j_0}$  the trial functions are linear in  $\xi, \eta$  for  
linear in  $x, y$  for

For  $\underline{j \leq j_0 - 1}$  all the elements belonging to  $(i, j)$  are  
ordinary triangles.

A) Interior Points

1)  $j = j_0$



$$\frac{\partial J}{\partial u_{i,j}}(u, v, \lambda) = 0 \Rightarrow$$

$$\begin{aligned} & 4 \cdot \left(\frac{3}{4}h\right)^{4/3} \left[ (u_{i,j+1} + u_{i-1,j+1}) \cdot \frac{3}{14}(j^{7/3} + (j+1)^{7/3} + \frac{3}{5}j^{10/3} - \frac{3}{5}(j+1)^{10/3}) \right. \\ & + (u_{i+1,j} + u_{i-1,j}) \cdot \frac{1}{2} \left( -\frac{1}{5}j^{4/3} - \frac{1}{30}(j-1)^{4/3} - \frac{1}{10}j^{2/3}(j-1)^{2/3} - \frac{3}{7}j^{7/3} \right. \\ & \left. \left. - \frac{9}{35}j^{10/3} - \frac{27}{455}j^{13/3} + \frac{27}{455}(j+1)^{13/3} \right) \right] \end{aligned}$$



$$+ u_{i,j} \left( -\frac{3}{10}j^{4/3} - \frac{2}{15}(j-1)^{4/3} - \frac{7}{30}j^{2/3} - \frac{3}{7}j^{7/3} \right. \quad (1a)$$

$$\left. + \frac{9}{35}(j+1)^{10/3} + \frac{27}{455}j^{13/3} - \frac{27}{455}(j+1)^{13/3} \right)$$

$$+ (u_{i+1,j-1} + u_{i,j-1}) \frac{1}{12} \cdot (j^{4/3} - (j-1)^{4/3})]$$

$$= (\lambda_{i-1,j+1} - \lambda_{i,j-1}) + (\lambda_{i,j+1} - \lambda_{i+1,j-1})$$

$$\frac{\partial \bar{J}}{\partial v_{i,j}} (u, v, \lambda) = 0 \Rightarrow$$

$$(v_{i,j+1} + v_{i-1,j+1}) \left( \frac{3}{5}j^{5/3} + \frac{3}{5}(j+1)^{5/3} + \frac{9}{20}j^{8/3}(j+1)^{8/3} \right)$$

$$+ (v_{i+1,j} + v_{i-1,j}) \left( -\frac{1}{4}j^{2/3} - \frac{1}{12}(j-1)^{2/3} - \frac{3}{5}j^{5/3} - \frac{9}{20}j^{8/3} - \frac{27}{220}j^{11/3} \right)$$

$$+ \frac{27}{220}(j+1)^{11/3}$$

$$+ v_{i,j} \left( -\frac{5}{6}j^{2/3} - \frac{1}{2}(j-1)^{2/3} - \frac{6}{5}j^{5/3} + \frac{9}{10}(j+1)^{8/3} - \frac{27}{110}(j+1)^{11/3} + \frac{27}{110}j^{11/3} \right)$$

$$(1b)$$

$$+ (v_{i,j} + v_{i+1,j-1}) \left( \frac{1}{6}j^{2/3} - \frac{1}{6}(j-1)^{2/3} \right)$$

$$= \frac{1}{h} [ (\lambda_{i,j+1} - \lambda_{i-1,j+1}) \left( \frac{3}{5}j^{5/3} + \frac{3}{5}(j+1)^{5/3} + \frac{9}{20}j^{8/3} - \frac{9}{20}(j+1)^{8/3} \right)$$

$$+ (\lambda_{i+1,j} - \lambda_{i-1,j}) \left( -\frac{1}{3}j^{2/3} - \frac{1}{6}(j-1)^{2/3} - \frac{3}{5}j^{5/3} - \frac{9}{40}j^{8/3} + \frac{9}{40}(j+1)^{8/3} \right)$$

$$+ (\lambda_{i+1,j-1} - \lambda_{i,j-1}) \left( \frac{1}{6} j^{2/3} - \frac{1}{6} (j-1)^{2/3} \right) ]$$

$$\frac{\partial \bar{J}}{\partial \lambda_{i,j}} (u, v, \lambda) = 0 \Rightarrow$$

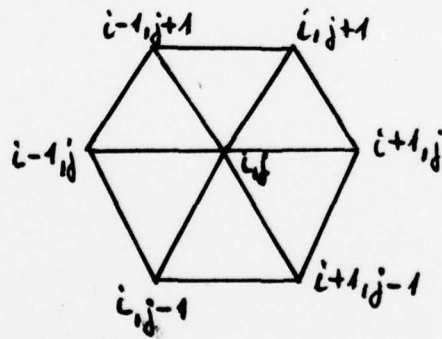
$$\frac{4 \cdot \left(\frac{3}{4}\right)^{2/3}}{h^{1/3}} [(v_{i,j+1} - v_{i-1,j+1}) \cdot \frac{3}{5} (j^{5/3} + (j+1)^{5/3} + \frac{3}{4} j^{8/3} - \frac{3}{4} (j+1)^{8/3})$$

$$+ (v_{i+1,j} - v_{i-1,j}) \left( -\frac{1}{3} j^{2/3} - \frac{1}{6} (j-1)^{2/3} - \frac{3}{5} j^{5/3} - \frac{9}{40} j^{8/3} + \frac{9}{40} (j+1)^{8/3} \right) \quad (1c)$$

$$+ (v_{i+1,j-1} - v_{i,j-1}) \left( \frac{1}{6} \cdot (j^{2/3} - (j-1)^{2/3}) \right) ]$$

$$= (u_{i-1,j+1} - u_{i,j-1}) + (u_{i,j+1} - u_{i+1,j-1})$$

$$2) \quad \underline{1 \leq j \leq j_0 - 1}$$



$$\frac{\partial \bar{J}}{\partial u_{i,j}} (u, v, \lambda) = 0 \Rightarrow$$

$$\frac{\left(\frac{3}{4}h\right)^{4/3}}{15} [u_{i,j} ((j+1)^{2/3} - (j-1)^{2/3}) (8(j+1)^{2/3} + 8(j-1)^{2/3} + 14j^{2/3})$$

$$+ (u_{i+1,j} + u_{i-1,j}) ((j+1)^{2/3} - (j-1)^{2/3}) ((j+1)^{2/3} + (j-1)^{2/3} + 3j^{2/3}) \quad (2a)$$

$$\begin{aligned}
 & + (u_{i,j+1} + u_{i-1,j+1}) \cdot 5((j+1)^{4/3} - j^{4/3}) \\
 & + (u_{i+1,j-1} + u_{i,j-1}) 5(j^{4/3} - (j-1)^{4/3})] \\
 & = (\lambda_{i-1,j+1} - \lambda_{i,j-1}) + (\lambda_{i,j+1} - \lambda_{i+1,j-1})
 \end{aligned}$$

$$\frac{\partial \bar{J}}{\partial v_{i,j}} (u, v, \lambda) = 0 \Rightarrow$$

$$\begin{aligned}
 & (3v_{i,j} + \frac{1}{2}v_{i+1,j} + \frac{1}{2}v_{i-1,j})((j+1)^{2/3} - (j-1)^{2/3}) \\
 & + (v_{i,j+1} + v_{i-1,j+1})((j+1)^{2/3} - j^{2/3}) \\
 & + (v_{i,j-1} + v_{i+1,j-1})(j^{2/3} - (j-1)^{2/3}) \\
 & = \frac{1}{h} [(\lambda_{i,j+1} - \lambda_{i-1,j+1})((j+1)^{2/3} - j^{2/3}) \\
 & + (\lambda_{i+1,j} - \lambda_{i-1,j})((j+1)^{2/3} - (j-1)^{2/3}) \\
 & + (\lambda_{i+1,j-1} - \lambda_{i,j-1})(j^{2/3} - (j-1)^{2/3})]
 \end{aligned} \tag{2b}$$

$$\frac{\partial \bar{J}}{\partial \lambda_{i,j}} (u, v, \lambda) = 0 \Rightarrow$$

$$\frac{2}{3} \cdot \frac{(\frac{3}{4})^{2/3}}{h^{1/3}} [(v_{i,j+1} - v_{i-1,j+1})((j+1)^{2/3} - j^{2/3})$$

$$+ (v_{i+1,j} - v_{i-1,j})((j+1)^{2/3} - (j-1)^{2/3}) \quad (2c)$$

$$+ (v_{i+1,j-1} - v_{i,j-1})(j^{2/3} - (j-1)^{2/3})]$$

$$= (u_{i-1,j+1} - u_{i,j-1}) + (u_{i,j+1} - v_{i+1,j-1})$$

Taylor expansion gives that the order of the equation is:

$$O(h^2) \quad \text{for } j \leq j_0 - 1$$

$$O(h) \quad \text{for } j = j_0$$

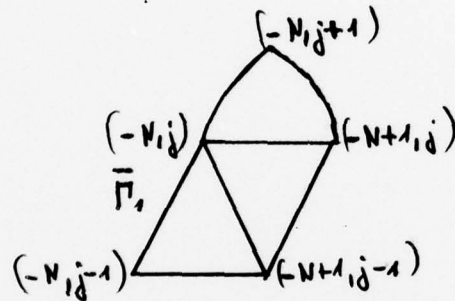
for sufficiently large values of  $j$ . For small values of  $j$  the accuracy of the equations can be less.

B) Boundary Points for the Cauchy problem

$$\bar{J}_1(u, v, \lambda) = \iint_{\bar{D}} [yu^2 - v^2 + \lambda(u_y - v_x)] dx dy + \int_{\bar{\Gamma}_1 \cup \bar{\Gamma}_2} \lambda(udx + vdy)$$

a) mesh points on  $\bar{\Gamma}_1$

1)  $j = j_0$



$$\frac{\partial}{\partial u_{-N, j}} \bar{J}_1(u, v, \lambda) = 0 \Rightarrow$$

$$\begin{aligned} & 6 \cdot \left(\frac{3}{4}h\right)^{4/3} [u_{-N, j+1} \left(\frac{3}{14}j^{7/3} + \frac{9}{35}j^{10/3} + \frac{9}{70}(j+1)^{10/3} + \frac{81}{910}j^{13/3} - \frac{81}{910}(j+1)^{13/3}\right) \\ & + u_{-N+1, j} \left(-\frac{1}{10}j^{4/3} - \frac{1}{60}(j-1)^{4/3} - \frac{1}{20}j^{2/3}(j-1)^{2/3} \right. \\ & \left. - \frac{3}{14}j^{7/3} - \frac{9}{70}j^{10/3} - \frac{27}{910}j^{13/3} + \frac{27}{910}(j+1)^{13/3}\right) \\ & + u_{-N, j} \left(\frac{-1}{10}j^{4/3} - \frac{1}{10}(j-1)^{4/3} - \frac{2}{15}j^{2/3}(j-1)^{2/3} \right. \\ & \left. - \frac{3}{7}j^{7/3} - \frac{9}{35}j^{10/3} - \frac{27}{455}j^{13/3} + \frac{27}{455}(j+1)^{13/3}\right) \\ & + u_{-N+1, j-1} \left(\frac{1}{12}j^{2/3} - \frac{1}{12}(j-1)^{2/3}\right) \end{aligned} \quad (3a)$$

$$+ u_{-N,j-1} \left( \frac{1}{30} j^{4/3} - \frac{1}{20} (j-1)^{4/3} + \frac{1}{60} j^{2/3} (j-1)^{2/3} \right) ]$$

$$= (\lambda_{-N,j+1} - \lambda_{-N+1,j-1}) + (\lambda_{-N,j} \frac{\lambda_{-N,j-1} + \lambda_{-N+1,j-1}}{2})$$

$$\frac{\partial}{\partial v_{-N,j}} \bar{J}_1(u,v,\lambda) = 0 \Rightarrow$$

$$v_{-N,j-1} \left( \frac{18}{5} j^{5/3} + \frac{27}{5} j^{8/3} + \frac{27}{10} (j+1)^{8/3} + \frac{243}{110} j^{11/3} - \frac{243}{110} (j+1)^{11/3} \right)$$

$$+ v_{-N+1,j} \left( -\frac{3}{2} j^{2/3} - \frac{1}{2} (j-1)^{2/3} - \frac{18}{5} j^{5/3} - \frac{27}{10} j^{8/3} - \frac{81}{110} j^{11/3} + \frac{81}{110} (j+1)^{11/3} \right)$$

$$+ v_{-N,j} \left( -2 j^{2/3} - 2 (j-1)^{2/3} - \frac{36}{5} j^{5/3} - \frac{27}{5} j^{8/3} - \frac{81}{55} j^{11/3} + \frac{81}{55} (j+1)^{11/3} \right) \quad (3b)$$

$$+ v_{-N+1,j-1} (j^{2/3} - (j-1)^{2/3})$$

$$+ v_{-N,j-1} \left( \frac{1}{2} j^{2/3} - \frac{1}{2} (j-1)^{2/3} \right)$$

$$= \frac{1}{h} [ (\lambda_{-N+1,j} - \lambda_{-N,j}) (-2 j^{2/3} - (j-1)^{2/3} - \frac{18}{5} j^{5/3} - \frac{27}{10} j^{8/3} + \frac{27}{20} (j+1)^{8/3})$$

$$+ (\lambda_{-N+1,j-1} - \lambda_{-N,j-1}) (j^{2/3} - (j-1)^{2/3}) ]$$



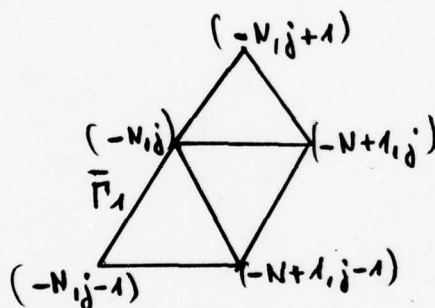
$$\frac{\partial}{\partial \lambda_{-N,j}} \bar{J}_1(u,v,\lambda) = 0 \Rightarrow \quad - 23 -$$

$$(u_{-N,j+1} - u_{-N+1,j-1}) + (u_{-N,j} - \frac{u_{-N,j-1} + u_{-N+1,j-1}}{2})$$

$$= \frac{(\frac{3}{4})^{2/3}}{h^{1/3}} [v_{-N+1,j} - v_{-N,j}] (-2j^{2/3} - (j-1)^{2/3} - \frac{18}{5}j^{5/3} - \frac{27}{20}j^{8/3} + \frac{27}{20}(j+1)^{8/3})$$

$$+ (v_{-N+1,j-1} - v_{-N,j-1}) (j^{2/3} - (j-1)^{2/3}) \quad (3c)$$

$$2) \quad \underline{1 \leq j \leq j_0 - 1}$$



$$\frac{\partial}{\partial u_{-N,j}} \bar{J}_1(u,v,\lambda) = 0 \Rightarrow$$

$$\frac{(\frac{3}{4}h)^{4/3}}{10} [u_{-N,j+1} (2(j+1)^{4/3} - 3j^{4/3} + j^{2/3}(j+1)^{2/3})$$

$$+ u_{-N+1,j} ((j+1)^{4/3} - (j-1)^{4/3} + 3j^{2/3}(j+1)^{2/3} - 3j^{2/3}(j-1)^{2/3})$$

$$+ 2u_{-N,j} ((j+1)^{4/3} + 3j^{4/3} - 3(j-1)^{4/3} + 3j^{2/3}(j+1)^{2/3} - 4j^{2/3}(j-1)^{2/3}) \quad (4a)$$

$$+ 5u_{-N+1,j-1} (j^{4/3} - (j-1)^{4/3})$$

$$\begin{aligned}
 & + u_{-N,j-1} (2j^{4/3} - 3(j-1)^{4/3} + j^{2/3}(j-1)^{2/3}) ] \\
 & = (\lambda_{-N,j+1} - \lambda_{-N+1,j-1}) + (\lambda_{-N,j} - \frac{\lambda_{-N,j-1} + \lambda_{-N+1,j-1}}{2})
 \end{aligned}$$

$$\frac{\partial}{\partial v_{-N,j}} \bar{J}_1(u,v,\lambda) = 0 \Rightarrow$$

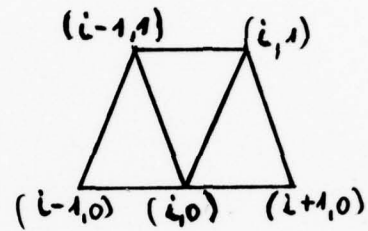
$$\begin{aligned}
 & \frac{1}{2} v_{-N,j+1} ((j+1)^{2/3} - j^{2/3}) + \frac{1}{2} v_{-N+1,j} ((j+1)^{2/3} - (j-1)^{2/3}) \\
 & + v_{-N,j} ((j+1)^{2/3} + j^{2/3} - 2(j-1)^{2/3}) + v_{-N+1,j-1} (j^{2/3} - (j-1)^{2/3}) \quad (4b)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} v_{-N,j-1} (j^{2/3} - (j-1)^{2/3}) \\
 & = \frac{1}{h} [ (\lambda_{-N+1,j} - \lambda_{-N,j}) ((j+1)^{2/3} - (j-1)^{2/3}) \\
 & + (\lambda_{-N+1,j-1} - \lambda_{-N,j-1}) (j^{2/3} - (j-1)^{2/3})
 \end{aligned}$$

$$\frac{\partial}{\partial \lambda_{-N,j}} \bar{J}_1(u,v,\lambda) = 0 \Rightarrow$$

$$\begin{aligned}
 & (u_{-N,j+1} - u_{-N+1,j-1}) + (u_{-N,j} - \frac{u_{-N,j-1} + u_{-N+1,j-1}}{2}) \\
 & = \frac{(\frac{3}{4})^{2/3}}{h^{1/3}} [ (v_{-N+1,j} - v_{-N,j}) ((j+1)^{2/3} - (j-1)^{2/3}) \\
 & + (v_{-N+1,j-1} - v_{-N,j-1}) (j^{2/3} - (j-1)^{2/3}) ] \quad (4c)
 \end{aligned}$$

b) mesh points on the parabolic line:  $j = 0$



$$i = -N+1, \dots, N-1$$

$$u_{i,0}$$

and

$$v_{i,0} \text{ are given}$$

$$\frac{\partial \bar{J}}{\partial \lambda_{i,0}} (u, v, \lambda) = 0 \Rightarrow$$

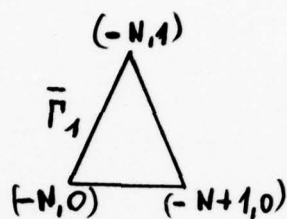
$$\frac{2 \cdot \left(\frac{3}{4}\right)^{2/3}}{h^{1/3}} [(v_{i,1} - v_{i-1,1}) + (v_{i+1,0} - v_{i-1,0})]$$

$$= 3(u_{i,1} + u_{i-1,1}) - (4u_{i,0} + u_{i-1,0} + u_{i+1,0})$$

This scheme is consistent and the order of the equation is:

$$u_y = v_x + O(h^{2/3})$$

$$(i,j) = (-N,0)$$



$$u_{-N,0}$$

and

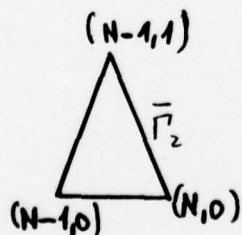
$$v_{-N,0} \quad \text{are given}$$

$$\frac{\partial \bar{J}}{\partial \lambda_{-N,0}} (u,v,\lambda) = 0 \rightarrow$$

$$\frac{(\frac{3}{4})^{2/3}}{h^{1/3}} (v_{-N+1,0} - v_{-N,0}) = (u_{-N,1} - \frac{u_{-N,0} + u_{-N+1,0}}{2})$$

$$u_y = v_x + O(h^{2/3})$$

$$(i,j) = (N,0)$$



$$u_{N,0} \text{ and}$$

$$u_{N,0} \text{ is given}$$

$$\frac{\partial \bar{J}}{\partial \lambda_{N,0}}(u, v, \lambda) = 0 \Rightarrow$$


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$$\left(\frac{3}{4}\right)^{2/3} \frac{1}{h^{1/3}} (v_{N,0} - v_{N-1,0}) = (u_{N-1,1} - \frac{u_{N-1,0} + u_{N,0}}{2})$$

$$u_y = v_x + O(h^{2/3})$$

Comparison of the Approximations

type of approximation j	linear in $\xi, \eta$ in all the elements	linear in $x, y$ in all the elements	linear in $\xi, \eta$ for $y > y_j$ linear in $x, y$ for $y \leq y_j$
0	$u_y = 1.2v_x + 0(h^{2/3})$	$u_y = v_x + 0(h^{2/3})$	$u_y = v_x + 0(h^{2/3})$
1	$u_y = 0.88v_x + 0(h^{2/3})$	$u_y = v_x + 0(h^{2/3})$	$u_y = 1.014v_x + 0(h^{2/3})$

One can see that if we take  $j_0 = 1$  we can already get a nearly consistent scheme:

$$u_y = 1.014v_x + 0(h^{2/3}) \quad \text{for } j = j_0 = 1$$

So only for very small values of  $h$  for which  $h^{2/3} \ll 0.01$  it is worthwhile to increase the value of  $j_0$ .



# 5. Cauchy problem for the Tricomi equation-Local Formulation

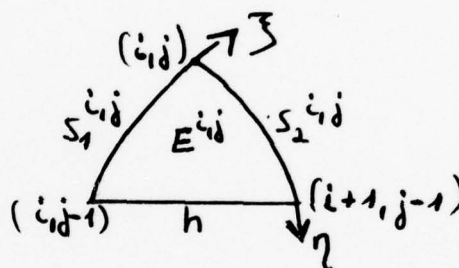
## A) Triangular Isoparametric Elements

We divide the domain into triangular isoparametric elements and take the same trial functions as before.

Let  $E^{i,j}$  be the element bounded by  $y = y_{j-1}$  and the two characteristics  $S_1^{i,j}$  and  $S_2^{i,j}$  which intersect in the mesh point  $(x_{i,j}, y_j)$ :

$$S_1^{i,j} : \eta = ih$$

$$S_2^{i,j} : \xi = (i+j)h$$



Step j ( $j = 1, \dots, 2N$ )

We assume that the solution  $(u, v, \lambda)$  is known at step  $j-1$ , so the problem is well-posed in element  $E^{i,j}$  with the free boundaries  $S_1^{i,j}$  and  $S_2^{i,j}$ . We take the variational formulation in every single element

$$J^{i,j}(u, v, \lambda) = \iint_{E^{i,j}} [yu^2 - v^2 + \lambda(u_y - v_x)] dx dy$$

$$+ \int_{S_1^{i,j} \cup S_2^{i,j}} \lambda(udx + vdy) \quad i = -N, \dots, +N-J$$

and we assume that  $\lambda$  is kept fixed on the free boundaries.

$$\frac{\partial}{\partial u_{i,j}} J^{i,j}(u,v,\lambda) = 0 \Rightarrow$$

$$\begin{aligned} & \left( \frac{3}{4}h \right)^{4/3} \cdot \frac{18}{7} [u_{i,j} (j^{7/3} - \frac{3}{5}(j-1)^{10/3} - \frac{6}{5}j^{10/3} - \frac{27}{65}(j-1)^{13/3} + \frac{27}{65}j^{13/3}) \\ & + (u_{i,j-1} + u_{i+1,j-1}) (\frac{1}{2}(j-1)^{7/3} + \frac{3}{5}(j-1)^{10/3} + \frac{3}{10}j^{10/3} + \frac{27}{130}(j-1)^{13/3} \\ & - \frac{27}{130}j^{13/3})] \\ & = \lambda_{i,j} - \frac{\lambda_{i,j-1} + \lambda_{i+1,j-1}}{2} \end{aligned} \quad (1a)$$

$$\frac{\partial}{\partial v_{i,j}} J^{i,j}(u,v,\lambda) = 0 \Rightarrow$$

$$\begin{aligned} & v_{i,j} (2j^{5/3} - \frac{3}{2}(j-1)^{8/3} - 3j^{8/3} - \frac{27}{22}(j-1)^{11/3} + \frac{27}{22}j^{11/3}) \\ & + (v_{i,j-1} + v_{i+1,j-1}) ((j-1)^{5/3} + \frac{3}{2}(j-1)^{8/3} + \frac{3}{4}j^{8/3} + \frac{27}{44}(j-1)^{11/3} - \frac{27}{44}j^{11/3}) \\ & = \frac{1}{h} (\lambda_{i+1,j-1} - \lambda_{i,j-1}) ((j-1)^{5/3} + j^{5/3} - \frac{3}{4}j^{8/3} + \frac{3}{4}(j-1)^{8/3}) \end{aligned} \quad (1b)$$

$$\frac{\partial}{\partial \lambda_{i,j}} J^{i,j}(u,v,\lambda) = 0 \Rightarrow$$

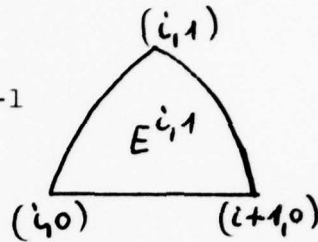
$$\begin{aligned} u_{i,j} - \frac{u_{i,j-1} + u_{i+1,j-1}}{2} &= \frac{18}{5} \cdot \left( \frac{3}{4} \right)^{2/3} \frac{1}{h^{1/3}} (v_{i+1,j-1} - v_{i,j-1}) (j^{5/3} + (j-1)^{5/3} \\ &+ \frac{3}{4}(j-1)^{8/3} - \frac{3}{4}j^{8/3}) \end{aligned} \quad (1c)$$

The solution of step  $j-1$  gives immediately  $v_{i,j}$  (1b) and  $u_{i,j}$  (1c) and substituting  $u_{i,j}$  into (1a) we get  $\lambda_{i,j}$ .

Step 0: the parabolic line

$$i = -N, \dots, N-1$$

$u_{i,0}$  and  $v_{i,0}$  are given.



$\lambda_{i,0}$  is not known.

$$\frac{\partial}{\partial \lambda_{i,0}} J^{i,1}(u, v, \lambda) = 0 \Rightarrow$$

$$u_{i,1} - \frac{u_{i,0} + u_{i+1,0}}{2} = \frac{27}{20} \frac{(\frac{3}{4}h)^{2/3}}{h^{1/3}} (v_{i+1,0} - v_{i,0}) \quad (li)$$

On the other side (lc) for  $j = 1$  will give

$$\frac{\partial}{\partial \lambda_{i,1}} J^{i,1}(u, v, \lambda) = 0 \Rightarrow$$

$$u_{i,1} - \frac{u_{i,0} + u_{i+1,0}}{2} = \frac{9}{10} \frac{(\frac{3}{4}h)^{2/3}}{h^{1/3}} (v_{i+1,0} - v_{i,0}) \quad (lii)$$

$u_{i,0}, u_{i+1,0}, v_{i,0}, v_{i+1,0}$  are known, so (li) and (lii) give a different value for  $u_{i,1}$  and  $\lambda_{i,0}$  is still unknown. So we cannot take the variation of  $\lambda$  on the parabolic line and must get the values of  $\lambda_{i,0}$  in another way:

The analytic connection between  $\lambda$  and  $v$  on the parabolic line is:

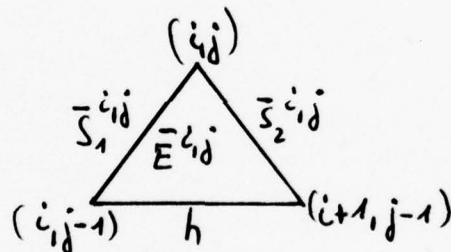
$$2v = \lambda_x.$$

So we can get the values of  $\lambda_{i,0}$  by numerical or analytic integration, because we know the values of  $v_{i,0}$ .

B) Linear elements

For  $j \geq j_0$  the elements  $E^{i,j}$  will be as before (triangular isoparametric and the trial function linear in  $\xi, \eta$ ).

For  $j < j_0$  the elements  $E^{i,j}$  will be ordinary triangles bounded by  $y = y_{j-1}$  and the linear approximation of the characteristics  $\bar{S}_1^{i,j}$  and  $\bar{S}_2^{i,j}$  which intersect in the mesh point  $(x_{i,j}, y_j)$ .



Step j ( $j = 1, \dots, j_0$ )

We assume that the solution  $(u, v, \lambda)$  is known at step  $j-1$ .

We take the variational formulation in every single element

$$J^{i,j}(u, v, \lambda) = \iint_{E^{i,j}} [yu^2 - v^2 + \lambda(u_y - v_x)] dx dy + \int_{\bar{S}_1^{i,j} \cup \bar{S}_2^{i,j}} \lambda(u dx + v dy)$$

$$i = -N, \dots, +N-j$$

and we assume that  $\lambda$  is kept fixed on the free boundaries.

$$\frac{\partial}{\partial u_{i,j}} \bar{J}^{i,j}(u,v,\lambda) = 0 \Rightarrow$$

$$\begin{aligned} & \left(\frac{3}{4}h\right)^{2/3} \cdot (j^{2/3} - (j-1)^{2/3}) \cdot \frac{1}{10} [u_{i,j} (6j^{2/3} + 4(j-1)^{2/3}) \\ & + (u_{i,j-1} + u_{i+1,j-1}) (2j^{2/3} + 3(j-1)^{2/3})] \\ & = \lambda_{i,j} - \frac{\lambda_{i,j-1} + \lambda_{i+1,j-1}}{2} \end{aligned} \quad (2a)$$

$$\frac{\partial}{\partial v_{i,j}} \bar{J}^{i,j}(u,v,\lambda) = 0 \Rightarrow$$

$$2v_{i,j} + v_{i,j-1} + v_{i+1,j-1} = \frac{2}{h} (\lambda_{i+1,j-1} - \lambda_{i,j-1}) \quad (2b)$$

$$\frac{\partial}{\partial \lambda_{i,j}} \bar{J}^{i,j}(u,v,\lambda) = 0 \Rightarrow$$

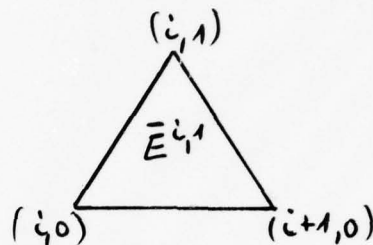
$$u_{i,j} - \frac{u_{i,j-1} + u_{i+1,j-1}}{2} = \frac{\left(\frac{3}{4}\right)^{2/3}}{h^{1/3}} (j^{2/3} - (j-1)^{2/3}) (v_{i+1,j-1} - v_{i,j-1}) \quad (2c)$$

We get again 3 explicit equations for  $u_{i,j}$ ,  $v_{i,j}$ ,  $\lambda_{i,j}$ .

Step 0: The parabolic line

$u_{i,0}$  and  $v_{i,0}$  are given

$$i = -N, \dots, N-1$$



$$\frac{\partial}{\partial \lambda_{i,0}} \bar{J}^{i,1}(u, v, \lambda) = 0 \Rightarrow$$

$$u_{i,1} - \frac{u_{i,0} + u_{i+1,0}}{2} = \frac{(\frac{3}{4})^{2/3}}{h^{1/3}} (v_{i+1,0} - v_{i,0}) \quad (2i)$$

On the other side (2c) for  $j = 1$  will give:

$$\frac{\partial}{\partial \lambda_{i,1}} \bar{J}^{i,j}(u, v, \lambda) = 0 \Rightarrow$$

$$u_{i,1} - \frac{u_{i,0} + u_{i+1,0}}{2} = \frac{(\frac{3}{4})^{2/3}}{h^{1/3}} (v_{i+1,0} - v_{i,0}) \quad (2ii)$$

which is the same scheme as (2i).

$\frac{\partial}{\partial \lambda_{i,0}} \bar{J}^{i,1}$  and  $\frac{\partial}{\partial \lambda_{i,1}} \bar{J}^{i,1}$  give the same scheme because the trial functions are linear in  $x, y$  in every element so  $u_y$  and  $v_x$  are constant in every triangle.

In order to get the values of  $\lambda_{i,0}$  we integrate the equation  $2v = \lambda_x$  as in the case of the isoparametric elements.



### Numerical Results

In the step-by-step method we must store only  $3 \cdot (2N+1)$  values, where  $2N+1$  is the number of points at the parabolic line. So we can use very small values of  $h$  and get any accuracy of the solution we want in condition that the solution converges.

Numerical examples are given in figures 2, 3 where the results of the following methods are compared:

1) The analytic solution

$$\text{Figure 2: } u(x,y) = \sinh x \left( y + \sum_{n=1}^{\infty} \frac{y^{3n+1}}{(3n+1)3n(3n-2)\dots 3} \right)$$

$$v(x,y) = \cosh x \left( 1 + \sum_{n=1}^{\infty} \frac{y^{3n}}{3n(3n-2)\dots 4 \cdot 3} \right)$$

$$\lambda(x,y) = 2\sinh x \left( 1 + \sum_{n=1}^{\infty} \frac{y^{3n}}{3n(3n-2)\dots 4 \cdot 3} \right)$$

$$\text{Figure 4: } u(x,y) = 6x$$

$$v(x,y) = 3y^2$$

$$\lambda(x,y) = 6xy^2$$

2) Solution of the Cauchy problem in the total hyperbolic region where the elements are isoparametric rectangles bounded by the curves  $\xi = \text{const}$  and  $\eta = \text{const}$  and one row of isoparametric triangles adjacent to the parabolic line, and the trial functions are bilinear in  $\xi, \eta$  in the isoparametric rectangles and linear in  $\xi, \eta$  in the isoparametric triangles. This method is demonstrated

by Yaniv [1].

3) Step-by-step method with isoparametric triangles and trial functions which are linear in  $\xi, \eta$  i.e.  $j_0 = 0$ .

4) Step-by-step method with linear corrections near the parabolic line.  $j_0 = 1$  was taken, i.e. in one row the trial functions are linear in  $x, y$  and in the others linear in  $\xi, \eta$ .

5) Step-by-step method with trial functions linear in  $x, y$  in the whole hyperbolic region, i.e.  $j_0 = 2N$ .

Initial values are given for  $u, v$  and  $\lambda$  in all the methods described above.

In both examples every second step is noted ( $j=0, 2, 4, \dots, 12$ ).

y	v	v-analytic= $3y^2$	$h=\frac{1}{6}$	$h=\frac{1}{12}$	$h=\frac{1}{24}$	$h=\frac{1}{48}$
0.25	0.1875		0.000	0.149	0.166	0.176
0.397	0.472		0.375	0.418	0.443	0.457
0.52	0.811		0.570	0.745	0.776	0.7925
0.63	1.191		1.05	1.15	1.15	1.17
0.731	1.603		1.33	1.52	1.559	1.580
0.825	2.044		1.878	1.955	1.997	2.02
0.915	2.511		2.211	2.415	2.460	2.485
1.00	3.000		2.811	2.899	2.947	2.9725
1.08	3.510		3.190	3.404	3.455	3.481
1.16	4.040		3.831	3.929	3.982	4.01
1.23	4.587		4.248	4.47	4.527	4.556
1.31	5.151		4.926	5.03	5.08	5.12

Table 1. analytic solution  $u = 6x$

$$v = 2y^2$$

$$\lambda = 6xy^2$$

The table shows the numerical results for  $v$  for different values of  $h$  in the step-by-step method with trial functions which are linear in  $x$  and  $y$ .

Convergence  $O(h)$  is demonstrated

$j \backslash v$	$v\text{-analytic}=3y^2$	Cauchy in every triangle linear in $\xi, \eta$	Cauchy in every triangle linear in $\xi, \eta(x, y \text{ for } j=1)$	Cauchy in every triangle linear in $x, y$
0	0.000	0.000	0.000	0.000
11	0.1875	0.000	0.000	0.000
2	0.472	0.395	0.384	0.375
3	0.811	0.564	0.565	0.570
4	1.191	1.075	1.064	1.053
5	1.603	1.320	1.321	1.329
6	2.044	1.902	1.891	1.878
7	2.511	2.200	2.201	2.211
8	3.000	2.837	2.825	2.811
9	3.510	3.178	3.178	3.190
10	4.040	3.858	3.846	3.831
11	4.587	4.235	4.236	4.248
12	5.151	4.954	4.942	4.926

Table 2: Numerical results of  $v$  for  $j=0,1,2,\dots,12$ ,

$$h = \frac{1}{6}$$

analytic solution:  $u = 6x$

$$v = 3y^2$$

$$\lambda = 6xy^2$$

Nonuniform (i.e. oscillating) convergence is seen.







#### FINAL REMARKS

Tricomi's problem, with a view of application to transonic flows and nonlinear generalizations, has induced much ingenious effort, with the progress attained ever increasing. A few problems, however, need further clarification. Unresolved problems exist regarding the local behavior of approximations near the parabolic line, the local behavior of weak and strong discontinuities (is a local loss of accuracy inherently unavoidable?) and the possible pollution of the whole field by local errors at boundaries and non-smooth regions. Further clarification is needed with regard to observed local numerical phenomena (e.g. decoupling of points, local under or super convergence and accuracy) and to a better understanding of numerical procedures at the parabolic (sonic) and discontinuity (shock waves) lines. Possible improvements are sought, trying alternatives to finite difference methods (e.g. Murman-Cole) and classical weak solutions (e.g. Friedrichs) via new formulations as the one proposed here, and finite elements following the inherent structure of the mathematical field (grid of characteristics, where real).

The results above are rough and exploratory. In view of new ways, further work and the generality of the underlying formulations, both analytic (variational) and discrete (characteristic grid with local treatment where needed) are indicated, addressing the questions above. Extensions at work are to 3-dimensional linear Tricomi-type problems, and to nonlinear mixed type equations.

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ENTROPY FUNCTIONS OF CONSERVATION LAWS OF MIXED TYPE

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## I. Introduction

Systems of nonlinear conservation laws, of the form

$$(1.1) \quad u_t + f(u)_x = 0$$

with  $u, f$   $n$ -dimensional vectors, are said to be hyperbolic in a region  $H$  (of  $u$ -space) if for every  $u \in H$ , the eigenvalues of  $f_u(u)$  are real and distinct. For example, in the system

$$(1.2) \quad \begin{cases} p_t - q_x = 0 \\ q_t - (p^2)_x = 0, \end{cases}$$

$H$  is the half-plane  $p > 0$ . The system (1.2) will be adopted on a prototype problem for the present discussion, as it arises in various applications [14, 15]. In the case  $p > 0$ , the uniqueness of the solution of the Cauchy problem for (1.2) is obtained in [15].

Some systems of conservation laws admit an entropy function, i.e. a convex scalar function  $U(u)$  such that [6,8]

$$(1.3) \quad f_u^T U_u = F_u$$

for another scalar function  $F$ . This is true in particular for pairs of equations and for some larger systems, such as the equations of compressible fluid flow. Multiplying (1.1) by  $U_u$  and using (1.8), it follows that smooth solutions of (1.1) satisfy an additional conservation law

$$(1.4) \quad U(u)_t + F(u)_x = 0.$$

For a given system one typically looks for functions  $U, F$  satisfying (1.4). Alternatively, a differential equation for  $U$  may be derived from (1.3). For example, multiplying the first equation in (1.2) by  $p^2$  and the second by  $q$ , it follows that (1.4) holds with

$$(1.5) \quad \underline{U}(p,q) = p^3/3 + q^2/2, \quad \underline{F}(p,q) = -p^2q.$$

We emphasize that (1.4) does not hold in general for weak solution of (1.1). Nevertheless, there are (for our purposes) two important implications of the existence of an entropy function. One is the existence of a change of variables, given by [2]

$$(1.6) \quad v = U_u(u)$$

which puts the system (1.1) into generalized symmetric form,  
[3]:

$$(1.7) \quad (\Phi_v(v))_t + (\Psi_v(v))_x = 0$$

where  $\Phi, \Psi$  are scalar functions of  $v$ . In fact,  $\Phi, \Psi$  are given explicitly by

$$(1.8) \quad \Phi(v) = u \cdot U_u(u) - U(u), \quad \Psi(v) = f(u) \cdot U_u(u) - F(u);$$

assuming that all the indicated derivatives exist, then differentiation of (1.8) shows that

$$(1.9) \quad u = \Phi_v(v), \quad f(u) = \Psi_v(v)$$

which establishes the equivalence of (1.1) and (1.7), even for weak solutions. For our example (1.2, 1.5), we readily obtain

$$(1.10) \quad v = (\tilde{p}, \tilde{q})^T = (p^2, q)^T,$$

$$(1.11) \quad \Phi(v) = \frac{2}{3} \tilde{p}^{3/2} + \frac{1}{2} \tilde{q}^2, \quad \Psi(v) = -\tilde{p} \tilde{q},$$

and from (1.7), the corresponding symmetric system is

(1.12)

$$\begin{cases} (\tilde{p})_t^{1/2} - \tilde{q}_x = 0 \\ \tilde{q}_t - \tilde{p}_x = 0. \end{cases}$$

Even for smooth initial data, the solution of (1.1) is not unique; the convexity of  $U$  is used as a means of removing the nonuniqueness. We suppose that  $u$ , a weak solution of (1.1), is the limit as  $\epsilon \rightarrow 0^+$ , boundedly in measure (two-dimensional measure in the  $x, t$  plane) of the solution  $u_\epsilon$  of

(1.13)

$$u_t + f(u)_x = \epsilon u_{xx},$$

the solution of which is smooth. Multiply (1.13) by  $\phi U_u$ , where  $\epsilon \in C_0^\infty$  is a nonnegative test function of  $x, t$ , and integrate over  $x, t$ . After partial integration we obtain

$$-\iint U(u_\epsilon) \phi_t + f(u_\epsilon) \phi_x = -\epsilon \iint u_{\epsilon, x} \cdot (U_{uu}(u_\epsilon) u_{\epsilon, x}) \phi + \epsilon \iint U(u_\epsilon) \phi_{xx}.$$

Passing to the limit, we obtain, using  $U_{uu} \geq 0$ ,

(1.14)

$$U(u)_t + F(u)_x \leq 0$$

in the sense of distributions. When  $f$  is genuinely non-



linear [7] the entropy inequality (1.14) is equivalent to the classical entropy condition for weak solutions of (1.1) [9], at least in the small; the theory is extended to include strong shocks in [11, 13]. Such entropy inequalities are readily incorporated into difference schemes, including such schemes to automatically select the physically correct weak solution [5, 8, 10, 12, 13].

The convexity of  $U$  is also related to the hyperbolicity of (1.1), as shown by the following theorem.

Theorem 1.1: Suppose that the system (1.1) admits an entropy function  $U$ , convex for  $u \in H$ . Then the eigenvalues of  $f_u(u)$  are real for all  $u \in H$ .

Proof. Differentiating (1.6) with respect to  $u$  and (1.9) with respect to  $v$ , we have  $U_{uu}^{-1}(u) = \Phi_{vv}(v)$ , where  $v = U_u(u)$ , so that the local convexity of  $U$  with respect to  $u$  is equivalent to that of  $\Phi$  with respect to  $v$ . The eigenvalues  $\lambda$  of  $f_u(u)$  are also the eigenvalues of the mixed symmetric problem.

$$(1.15) \quad \Phi_{vv} r = \lambda \Phi_{vv} r$$

where  $r$  is the corresponding eigenvector. Since  $\Psi_{vv}$  is symmetric and  $\Phi_{vv}$  symmetric and positive definite, the eigenvalues  $\lambda$  are real. Thus when hyperbolicity fails in (1.1),

for example we allow  $p < 0$  in (1.2), the convexity of  $U$  fails, and (1.14) no longer follows from (1.12). In fact, we may question whether (1.12) is an appropriate form of regularization in this case. Even in the purely hyperbolic case, it is known that different choices of regularization can lead to different weak solutions as limits [1, 4, 16]. We can recover (1.14), even with  $U$  not necessarily convex, by using a regularized form of (1.7). Indeed, let  $v$  be the limit of  $v_\epsilon$ , satisfying

$$(1.16) \quad (\Phi_v(v_\epsilon))_t + (\Psi_v(v_\epsilon))_x = \epsilon(Av_{\epsilon,x})_x$$

where  $A$  is positive definite, symmetric;  $A$  may even depend on  $v_\epsilon$ . Multiplying (1.16) by  $\phi v_\epsilon$  and integrating, we use (1.6, 1.8, 1.9) to obtain

$$(1.17) \quad U(u) = v \cdot \Phi_v - \Phi, \quad F(u) = v \cdot \Psi_v - \Psi$$

and readily obtain (1.14) as above. However, when hyperbolicity fails, the transformation  $u \rightarrow v$ , described in our sample problem by (1.10), is in general no longer invertible, so that (1.1) and (1.7) are no longer equivalent. These problems may be aggravated in general by the existence of many entropy functions, each with its corresponding gen-

eralized symmetric system.

In order to approximate the physically relevant solutions of a given system (1.1) of mixed type, we require knowledge of how much accitional physics is needed to make the problem well-posed, in addition to the specification of proper initial/boundary conditions. For example, which dependent variables should be used, which form(s) of regularization are appropriate, and are entropy inequalities (1.14) sufficient? Obviously, these questions are not independent.

In section II, we briefly review the theory of piecewise smooth weak solutions, in slightly generalized form. In Section III, we apply these methods to a class of problems including (1.2), and obtain suitable forms of regularization and entropy conditions. These results are incorporated into a difference scheme for (1.2) in Section IV.

## II. Discontinuous solutions of systems of mixed type

We next consider systems of the form

$$(2.1) \quad g(u)_x + f(u)_y = 0, \quad u \in D = H \cup E \cup (\partial H \cap \partial E)$$

which are hyperbolic in  $H$  only. For example, we rewrite our model problem (1.2) in the form

$$(2.2) \quad \begin{cases} (p^2)_x - q_y = 0 \\ q_x - p_y = 0. \end{cases}$$

Indeed, we have in mind primarily the case that (2.1) is a pair of equations, and is elliptic in  $E$ . Thus in (2.2),  $H(E)$  is the half-plane  $p > 0$  ( $p < 0$ ).

A system (2.1) will admit an entropy function if there exists a vector  $w(u)$  such that

$$(2.3) \quad g_u^T(u)w(u) = U_u(u), \quad \text{and} \quad f_u^T(u)w(u) = F_u(u).$$

As previously, the existence of an entropy function implies that the system can be put in generalized symmetric form; setting

$$(2.4) \quad \Phi(w) = g(u) \cdot w(u) - U(u), \quad \Psi(w) = f(u) \cdot w(u) - F(u),$$

it follows from (2.3) that

$$(2.5) \quad \Phi_w(w) = g(u) \quad \text{and} \quad \Psi_w(w) = f(u),$$

so that (2.1) is equivalent to

$$(2.6) \quad (\Phi_w(w))_x + (\Psi_w(w))_y = 0$$

as long as the mapping of  $D \rightarrow R$  determined by  $u \rightarrow w(u)$  is invertible. The convexity of  $U$  (with respect to  $u$ ) and of  $\Phi$  (with respect to  $w$ ) are no longer equivalent, however.

Our example (2.2) is already in symmetric form, with

$$(2.7) \quad u = (p, q)^T, \quad \Phi(u) = p^3/3 + q^2/2, \quad \Psi(u) = -pq.$$

Thus we may set  $u = w$  and find  $U, F$  from (2.4), obtaining

$$(2.8) \quad U(u) = \frac{2}{3}p^3 + \frac{1}{2}q^2, \quad F(u) = -pq.$$

However, there are other possible choices, for example

$$(2.9) \quad w = (p^3 + \frac{3}{2}q^2, 3p^2q)^T, \quad \hat{U}(u) = \frac{2}{5}p^5 + \frac{3}{2}p^2q^2, \quad \hat{F}(u) = -p^3q - \frac{1}{2}q^3.$$

Corresponding to this choice of  $w$ , there is a symmetric form (2.6), with  $\Phi(w), \Psi(w)$  obtained from (2.4, 2.9). Thus for the pair (1.2) we have three possible entropy functions:  $\hat{U}$ , given by (1.5);  $U$ , given by (2.8); and  $\hat{U}$ , given by (2.9). More could be constructed.  $U$  and  $\hat{U}$  are convex in  $H$ ;  $\hat{U}$  is convex when  $p \geq (\frac{3}{4}q^2)^{1/3}$ . To each entropy function corresponds a choice of dependent variable (1.10), (2.7) of (2.9), and a corresponding symmetric system, two examples of which are (1.12) and (2.2). These systems are all equivalent in the region  $p \geq (\frac{3}{4}q^2)^{1/3}$ ; the two systems (1.12) and (2.2) are equivalent for  $u \in H$ , but not for all  $u \in D$ . To decide which, if any, of these systems can correspond to a well-posed problem, we consider weak solutions containing shocks.

For a specific choice of dependent variable, denoted by  $w$ , we consider our system in symmetric form (2.6). Let us assume that  $w$  has an isolated discontinuity, along a shock curve in the  $x, y$  plane, described by

$$(2.10) \quad -\sin\theta dx + \cos\theta dy = 0.$$

Equation (2.10) corresponds to  $s = \tan\theta$ , where  $s$  is



the shock speed, in the case where  $x$  is a time variable. However, in systems of mixed type, infinite shock speeds are possible; thus the polar representation.

The Rankine-Hugoniot relations, expressing conservation of material, are

$$(2.11) \quad -\sin\theta[\phi_w] + \cos\theta[\psi_w] = 0,$$

where  $[ ]$  denotes the jump in a quantity across a shock, as usual. At any point  $w$ , the characteristic speeds  $\tan \eta_i$  and corresponding eigenvectors  $r_i$ ,  $i = 1, \dots, n$ , are given by

$$(2.12) \quad \cos\eta_i \psi_{ww} r_i = \sin\eta_i \phi_{ww} r_i;$$

the system (2.6) is hyperbolic at  $w$  if the  $\eta_i$  are real and distinct, and elliptic at  $w$  if the  $\eta_i$  are all complex. The system (2.6) is hyperbolic whenever  $\phi$  is locally convex. Assuming that  $\phi$  is locally convex for all  $w \in H$ , and that  $H$  is a convex region (of  $w$ -space), shocks of infinite speed, corresponding to  $\cos\theta = 0$ ,  $[\phi_w] = 0$ , cannot connect two points in  $H$ , but such shocks might connect a point in  $H$  with one in  $E$ .

Given a point  $w_0$ , let  $\Gamma(w_0)$  denote the set of points in  $D$  which can be connected to  $w_0$  by a shock, i.e. for

which the Rankine-Hugoniot relations can be satisfied, for some value of  $\theta$ . By definition,  $\Gamma(w_0)$  does not contain  $w_0$  itself.  $\Gamma(w_0)$  is a family of one-parameter manifolds, described by the differential equation

$$(2.13) \quad \cos\sigma(\cos\sigma\psi_{ww}(w) - \sin\sigma\phi_{ww}(w))w' = \sigma'(\phi_w(w) - \phi_w(w_0)),$$

where primes denote differentiation along  $\Gamma(w_0)$ , and  $\sigma = \sigma(w, w_0)$  is the value of  $\theta$  for which the Rankine-Hugoniot relation (2.11) holds between  $w_0$  and  $w$ . We next assume that [11].

$$(2.14) \quad \sigma' \neq 0 \quad \text{for all } w_0 \in D, w \in \Gamma(w_0)$$

where  $\sigma'$  is determined from (2.13), with  $w' > 0$ . This is equivalent to the assumption that  $\tan\sigma(w, w_0)$  is never one of the characteristic speeds at  $w_0$  or at  $w$ . For  $w \in E$ , this condition is always satisfied, as the characteristic speeds are complex. Thus (2.14) is really only an assumption about the hyperbolic region.

The condition (3.14) was postulated in [11] as a statement of strong nonlinearity of the system (1.1), assumed to be hyperbolic. When (1.1) or (2.1) is hyperbolic, this condition implies the classical condition of genuine non-

linearity [7], and is almost equivalent for pairs of equations. For  $w_0 \in H$ ,  $\Gamma(w_0) \cap H$  consists of  $2n$  smooth distance one-parameter manifolds, each with  $w_0$  as one of its limit points, and the other limit point on the boundary of  $H$  or at infinity. Near  $w_0$ , the manifolds of  $\Gamma(w_0)$  are in the direction of  $\pm r_i(w_0)$ , depending on whether  $\sigma$  increases or decreases as one moves away from  $w_0$ .

For  $w_0 \in E$ , it follows from (2.13) that  $\Gamma(w_0)$  does not contain  $w_0$  as a limit point, for if it did, at least one of the characteristic speeds at  $w_0$  would have to be real. Thus there are no sufficiently weak shocks connecting two points in  $E$ .

For the system (2.1), the jump conditions (2.11) are

$$(2.15) \quad \begin{cases} \sin\theta(p^2 - p_0^2) + \cos\theta(q - q_0) = 0 \\ \sin\theta(q - q_0) + \cos\theta(p - p_0) = 0 \end{cases}$$

where  $w = (p, q)^T$ ,  $w_0 = (p_0, q_0)^T$ . From (2.15), we readily obtain,

$$(2.16) \quad \sin^2\theta(p + p_0) = \cos^2\theta,$$

so that  $p + p_0 \geq 0$ , i.e. there are no shocks connecting two

points in  $E$ . Below we show  $\Gamma(w_0)$  for  $w_0$  in different regions.

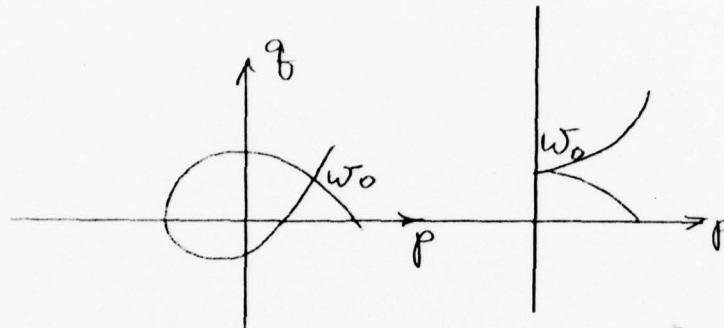
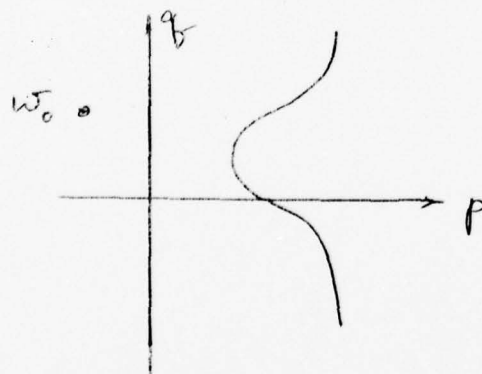


Fig. 1  $\Gamma(w_0)$  for  $w_0 \in H$  ( $p_0 > 0$ )

$\Gamma(w_0)$  for  $w_0 \in \partial H \cap \partial E$  ( $p_0 = 0$ )



$\Gamma(w_0)$  for  $w_0 \in E$  ( $p_0 < 0$ )

Corresponding to the symmetric system (2.6), there exists an entropy function

$$(2.17) \quad U(w) = w \cdot \phi_w - \phi, \quad F(w) = w \cdot \psi_w - \psi;$$

the entropy inequality (1.14) is equivalent to regaining that [8]

$$(2.18) \quad -\sin\theta[U] + \cos\theta[F] \leq 0.$$

For shocks which are limits of viscous profiles, obtained by regularizing (2.6) as in (1.13) or (1.16), (2.18) holds with strict inequality. In the simplest case, when the right side of (2.6) is replaced by a second directional derivative, the viscous profile between two states  $w_-, w_+$ , is the solution of the system of ordinary differential equations

$$(2.19) \quad \begin{aligned} \frac{dw}{dz} = & -\sin\sigma(w_1, w_0)(\phi_w(w(z)) - \phi_w(w_0)) + \\ & + \cos\sigma(w_+, w_-)(\psi_w(w(z)) - \psi_w(w_+)), \\ w(+\infty) = & w_+, \quad w(-\infty) = w_-. \end{aligned}$$

Let us examine the implications of (2.18) as applied to the system (2.2), utilizing each of our entropy functions. With  $U, F$  given by (2.8), we use the jump conditions (2.15) to obtain

$$(2.20) \quad -\sin\theta[U] + \cos\theta[F] = -\frac{1}{6}\sin\theta(p-p_0)^3$$

which is the entropy jump in a shock connecting two states  $(p_0, q_0)$  and  $(p, q)$ . Thus for this entropy function, we see that (2.18) implies that either that portion of  $\Gamma(p_0, q_0)$  to the right or to the left of the line  $p = p_0$  is admissible, depending on whether  $(p_0, q_0)$  is the upstream or downstream point.

The entropy function (1.5) was obtained with the variables  $x, y$  switched, which is equivalent to switching  $\hat{U}$  and  $\hat{F}$ . Therefore to compute the entropy jump in this case, we compute

$$(2.21) \quad -\sin\theta[\hat{F}] + \cos\theta[\hat{U}] = \frac{1}{6}\sin\theta(p-p_0)^2(q-q_0).$$

In this case, either that portion of  $\Gamma(p_0, q_0)$  above or below the line  $q = q_0$  is admissible, and possibly the shock corresponding to  $p = -p_0, q = q_0$ .

Finally for the entropy function  $\hat{U}$ , given by (2.9), we obtain (in the special case  $q_0 = 0$ )

$$(2.22) \quad -\sin\theta[\hat{U}] + \cos\theta[\hat{F}] = -\frac{1}{10}\sin\theta(p-p_0)^3(4p^2 + 2pp_0 - p_0^2),$$

and the curve  $\Gamma(p_0, q_0)$  is divided as shown in Fig. 2, the



solid or dashed portions admissible depending on whether  $(p_0, q_0)$  is upstream or downstream.

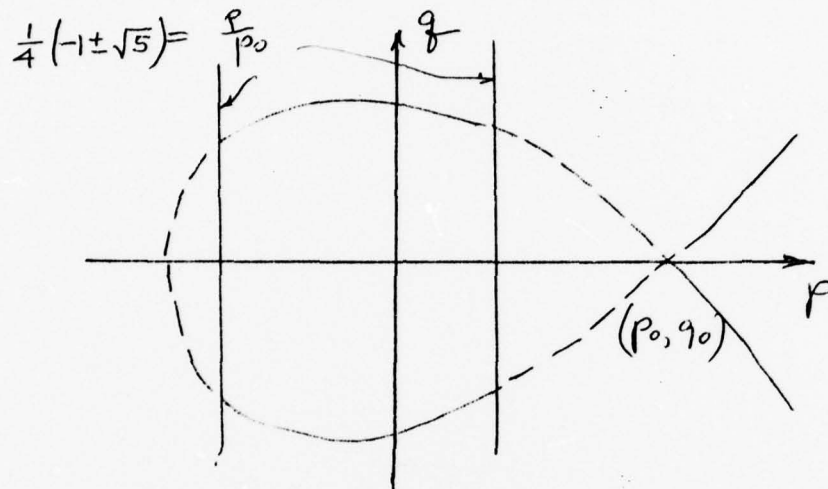


Fig. 2: Allowed shocks for entropy function  $\hat{U}$

### III. A class of pairs of mixed type.

We consider the class of pairs of equations

$$(3.1) \quad \begin{cases} \xi'(p)_x - q_y = 0 \\ q_x - p_y = 0 \end{cases}$$

which corresponds to (2.6) with  $u = (p, q)^T$

$$(3.2) \quad \Phi(p, q) = \xi(p) + q^2/2, \quad \Psi(p, q) = -pq$$

and admits a corresponding entropy function

$$(3.3) \quad U(p, q) = u \cdot \Phi_u - \Phi = p\xi'(p) - \xi(p) + q^2/2, \quad F(p, q) = -pq.$$

As in the sample problem (1.2), which is of course a special case of (3.1), there exist other functions  $\tilde{U}, \tilde{F}$  such that (1.14) holds, for example

$$(3.4) \quad U(p, q) = q\xi'(p), \quad F(p, q) = -\xi(p) - q^2/2$$

The system (3.1) is hyperbolic whenever  $\xi''(p) > 0$ ; the specific assumptions we make on  $\xi(p)$  are

$$(3.5) \quad \xi''(p) > 0, p > 0, \xi''(p) < 0, p < 0,$$

$$(3.6) \quad \xi'''(p) > 0, p \geq 0,$$

the assumption (3.6) assuring that the system is strongly nonlinear in the sense of [11] in the hyperbolic region. Without loss of generality, we take  $\xi(0) = \xi'(0) = 0$ .

The Rankine-Hugoniot relations (2.11), describing which states  $(p, q)$  can be connected to a given state  $(p_0, q_0)$  are in this case

$$(3.7) \quad \sin\theta(\xi'(p) - \xi'(p_0)) + \cos\theta(q - q_0) = 0.$$

$$(3.8) \quad \sin\theta(q - q_0) + \cos\theta(p - p_0) = 0.$$

From (3.7, 3.8), it is clear that there are no shocks with  $\theta = 0$ , but shocks between the elliptic and hyperbolic regions with  $\theta = \pm\pi/2$  may exist. From (3.2, 3.8), we easily obtain

$$(3.9) \quad \frac{\xi'(p) - \xi'(p_0)}{p - p_0} = \cot^2\theta \geq 0;$$

since the elliptic region  $(p < 0)$  is convex and  $\xi' < 0$  there, we have

Theorem 3.1: There are no shocks connecting two states in the elliptic region.

Let us therefore take  $(p_0, q_0) \in H$ ,  $q_0 = 0$  without loss of generality, and ask about the shape of the curve  $\Gamma(p_0, q_0)$ . From (3.6), we know that  $\theta' \neq 0$  (prime denoting differentiation along  $\Gamma$ ) on each branch of  $\Gamma(p_0, q_0)$ . Then from (3.8) we have

Theorem 3.2: Each branch of  $\Gamma(p_0, q_0)$  is star-shaped about the point  $p_0, q_0$ .

It follows from (3.6, 3.8, 3.9) that for  $p < p_0$

$$(3.10) \quad |\cot \theta| = \left| \frac{q - q_0}{p - p_0} \right| \leq (\xi''(p_0))^{1/2}$$

and then from Theorem 3.2 and (3.10)

Theorem 3.3: For  $(p_0, q_0) \in H$ , the two branches of  $\Gamma(p_0, q_0)$  initially in the direction of decreasing  $p$  must enter the elliptic region. These two branches will eventually join (as in Fig. 1 above) if  $\xi'(p) = \xi'(p_0)$  for some  $p < 0$ . If  $\xi'(p) < \xi'(p_0)$  for all  $p < 0$ , the two branches will remain apart, as shown in Fig. 3 below.

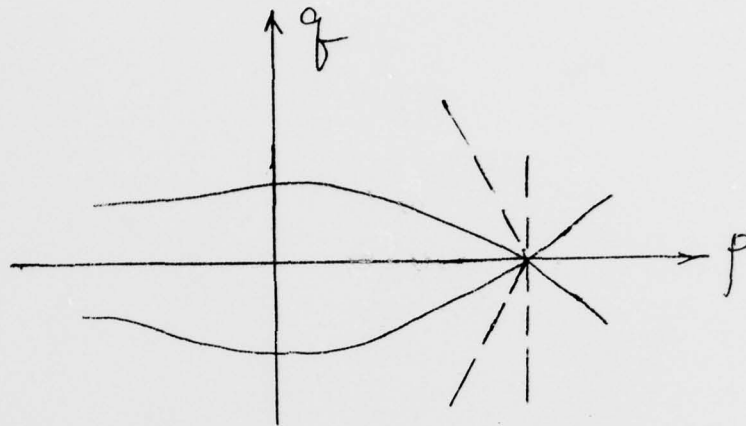


Fig. 3:  $\Gamma(p_0, q_0), (p_0, q_0) \in H, \xi'(p) < \xi'(p_0)$  for all  $p < 0$

The situation described in Fig. 3 can only arise for sufficiently large  $p_0$ ; for small positive  $p_0$ ,  $\Gamma(p_0, q_0)$  will be as shown in Fig. 1. For  $(p_0, q_0) \in E$  or on the boundary between  $E$  and  $H$ , the curves of  $\Gamma(p_0, q_0)$  resemble those of Fig. 1.

The entropy inequality (2.18), applied to (3.3), (3.4) respectively, gives

$$(3.11) \quad -\sin\theta[U] + \cos\theta[F] = \sin\theta[\xi(p) - \xi(p_0) - \frac{1}{2}(p - p_0)(\xi'(p) + \xi'(p_0))],$$

$$(3.12) \quad -\sin\theta[\tilde{U}] + \cos\theta[\tilde{F}] = \sin\theta(q - q_0) \left[ \frac{\xi(p) - \xi(p_0)}{p - p_0} - \frac{1}{2}(\xi'(p) + \xi'(p_0)) \right].$$

Exactly as in the sample problem (2.2), the entropy function  $U$  leads to a division of  $\Gamma(p_0, q_0)$  with respect to

$p-p_0$ , and the entropy function  $\tilde{U}$  to a division of  $\Gamma(p_0, q_0)$  with respect to  $q-q_0$ . Alternatively, the inequality (1.14) as applied to the entropy function (3.3) is equivalent for piecewise smooth solutions to the inequality

$$(3.13) \quad p_x < +\infty;$$

(1.14) with the entropy function (3.4) is equivalent to

$$(3.14) \quad q_y < +\infty.$$

It is quite natural that different entropy functions should give different admissible shocks, as they correspond to different forms of regularization of (3.1). The system (3.1) is already in the symmetric form corresponding to  $U$ , and so any regularization of the form (1.16) leads to (1.14). The symmetric form of (3.1) corresponding to  $U$  given (3.4) involves the transformation  $\tilde{p} = \xi'(p)$ ,  $\tilde{q} = q$ , which is not invertible when  $p$  can change sign. Alternatively, we could obtain (1.14) for  $\tilde{U}$  by a regularization of (3.1)

$$(3.15) \quad \begin{cases} \xi'(p)_x - q_y = q_{xx} \\ q_x - p_y = (\xi'(p))_{xx} \end{cases}$$



which may or may not be physically reasonable. Some knowledge of the physical dissipation mechanism is necessary to choose the correct form of regularization and/or entropy inequality. In view of the results of [1, 17] it seems likely that this is true even in the purely hyperbolic case. We conclude this section by proving that

Theorem 3.4: The shocks allowed by (2.18), (3.3) (alternatively (3.13)) are the limits of viscous profiles.

Proof: Let  $(p_+, q_+)$ ,  $(p_-, q_-)$  be two states which can be connected by a shock, with  $p_+ > p_-$ . Let  $\theta$  be such that (3.7), (3.8) are satisfied (between  $(p_+, q_+)$  and  $(p_-, q_-)$ ), with  $\sin\theta > 0$  for definiteness. We wish to show the existence of a solution of (2.19), which becomes

$$(3.16) \quad \begin{cases} p_z = -\sin\theta(\xi'(p) - \xi'(p_0)) - \cos\theta(q - q_+) \\ q_z = -\sin\theta(q - q_+) - \cos\theta(p - p_+) \end{cases}$$

$$p(\pm\infty) = p_{\mp}, \quad q(\pm\infty) = q_{\mp}$$

It follows from (3.8) that  $q_+ - q_-$  has the same sign on  $\cos\theta$ , which we take positive. Let  $\Omega$  be the rectangle with sides parallel to the  $p, q$  axes and  $(p_-, q_-)$ ,  $(p_+, q_+)$  at opposite corners, as shown in Fig. 4. From (3.16) the orbit

leaves  $\Omega$  on all four sides.

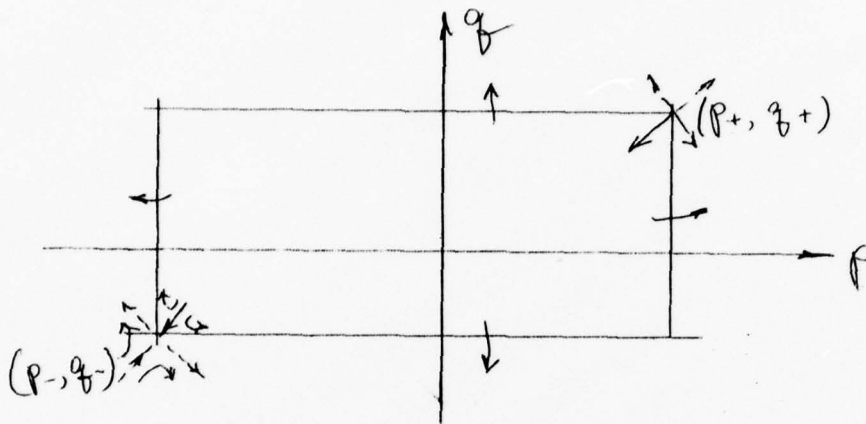


Fig. 4: Existence of viscous profile

Using the strong nonlinearity condition (3.6), it is not difficult to prove that  $(p_+, q_+)$  is an improper node and  $(p_-, q_-)$  is a saddle, with respect to the vector field generated by (3.16) [11]. This holds even if  $(p_-, q_-)$  is in the elliptic region. It is also easily shown that there are no other critical points in  $\Omega$ , so that the orbit entering the saddle from within  $\Omega$  must have come from  $(p_+, q_+)$ . This completes the proof, and also shows the uniqueness of the orbit up to translation, as desired.

#### IV. A difference scheme for the small disturbance problem

In this section we utilize the results above to construct a finite difference approximation to (2.1), which automatically excludes unphysical solutions and which approximates (2.1) to second order in the mesh size even when the solution is not smooth. The geometry, boundary conditions, and assumed form of the solution are shown in Fig. 5,

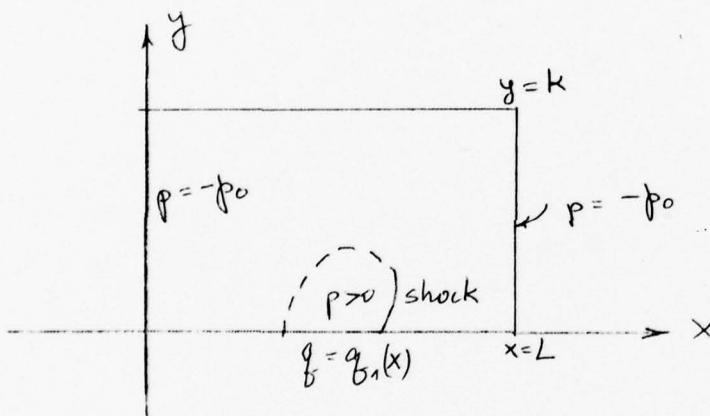


Fig. 5: Transonic flow problem

in which  $p_0 > 0$  and the smooth function  $q_1(x)$  are given. We assume that the solution is piecewise smooth, and of bounded variation in  $x$ , uniformly in  $y$ . The shock curve will reach the  $x$ -axis in general, but we assume that it does not reach the lines  $x = L$  or  $x = 0$ .

The discrete variables  $p_j^k, q_{j+1/2}^{k+1/2}$  are oriented as shown in Fig. 6. The  $q$ 's are oriented at the 0-points

and  $q_{j+1/2}^{k+1/2}$  is our

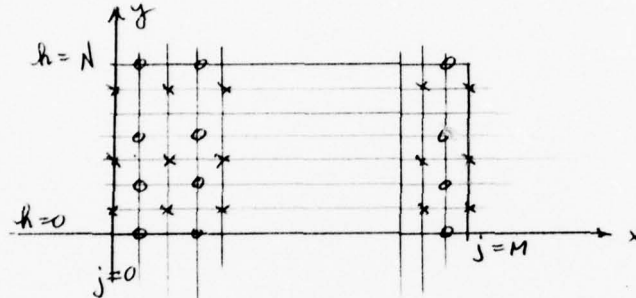


Fig. 6: Mesh point orientation

approximation to  $q(x,y)$ , for  $x,y$  in the rectangle  $j\Delta x < x < (j+1)\Delta x$ ,  $k\Delta y < y < (k+1)\Delta y$ . The  $p$ 's are oriented at the  $x$ -points in Fig. 6; however, their interpretation is more complicated. Let

$$(4.1) \quad \mu_{j+1/2}^k = \frac{1}{2}(p_j^k + p_{j+1}^k);$$

then in the rectangle  $(j-\frac{1}{2})\Delta x < x < (j+\frac{1}{2})\Delta x$ ,  $(k-\frac{1}{2})\Delta y < y < (k+\frac{1}{2})\Delta y$ , our approximation to  $p(x,y)$  is piecewise linear, given by

$$(4.2) \quad p_h(x,y) = \left[ \frac{(j+\frac{1}{2})\Delta x - x}{\Delta x} \right] \mu_{j-1/2}^k + \left[ \frac{x - (j-\frac{1}{2})\Delta x}{\Delta x} \right] \mu_{j+1/2}^k.$$

The difference approximations determining the  $p_j^k$ ,  $q_{j+1/2}^{k+1/2}$  are as follows:

$$(4.3) \quad \frac{p_{j+1/2}^{k+1/2} - p_{j-1/2}^{k+1/2}}{\Delta x} - \frac{p_j^{k+1} - p_j^k}{\Delta y} = 0, \quad j=1,2,\dots,M-1, \quad k=0,1,\dots,N-1;$$

$$(4.4) \quad \frac{(p_{j+1}^k)^2 - (p_j^k)^2}{\Delta x} - \frac{q_{j+1/2}^{k+1/2} - q_{j-1/2}^{k-1/2}}{\Delta y} = 0, \quad j=0, M-1, \quad k=0,1,\dots,N;$$

$$(4.5) \quad \frac{(\mu_{j+3/2}^k - \mu_{j-1/2}^k)(\mu_{j+3/2}^k + \mu_{j+1/2}^k + \mu_{j-1/2}^k)}{3\Delta x} - \frac{q_{j+1/2}^{k+1/2} - q_{j-1/2}^{k-1/2}}{\Delta y}$$

$$= -\frac{1}{\Delta x} [\mu_{j+5/2}^k \zeta_{j+5/2} - 4\mu_{j+3/2}^k \zeta_{j+3/2} +$$

$$+ 6\mu_{j+1/2}^k \zeta_{j+1/2} - 4\mu_{j-1/2}^k \zeta_{j-1/2} + \mu_{j-3/2}^k \zeta_{j-3/2}]$$

$$j = 1,2,\dots,M-2, \quad k = 0,1,\dots,N.$$

In (4.5),  $\zeta_{j+1/2} = \zeta((j+1/2)\Delta x)$ ,  $\zeta$  a nonnegative  $C_0^\infty$  function chosen so that  $\zeta > 0$  in a region containing the shock, and  $\zeta \equiv 0$  near the boundaries in  $x$ , so that (4.5) does not require any  $p$ -values outside the rectangle.

We first show that the difference scheme (4.3-4.5) satisfies a discrete form of the inequality (1.14) for the entropy function (2.8). Let  $\phi_j^k = \phi(j\Delta x, k\Delta y)$ ,  $\phi$  a nonnegative  $C_0^\infty$  test function; multiply (4.3) by

$$\Delta x \Delta y \left( \frac{q_{j+1/2}^{k+1/2} + q_{j-1/2}^{k+1/2}}{2} \right) \phi_j^{k+1/2}$$

and (4.5) by  $\Delta x \Delta y \mu_{j+1/2}^k \phi_{j+1/2}^k$ ; add and sum over  $j, k$ . We assume that the discrete variables are uniformly bounded, and use the smoothness of  $\zeta$  and the test function  $\phi$  repeatedly. After several partial summations, we obtain

$$\begin{aligned} & \Delta x \Delta y \sum_{j,k} \frac{(\mu_{j+1/2}^k)^3 + (\mu_{j+1/2}^k)^2 \mu_{j-1/2}^k + \mu_{j+1/2}^k (\mu_{j-1/2}^k)^2 + (\mu_{j-1/2}^k)^3}{6} \\ & \quad \left( \frac{\phi_{j+1/2}^k - \phi_{j-1/2}^k}{\Delta x} \right) \\ & + \frac{1}{2} (q_{j+1/2}^{k+1/2})^2 \left( \frac{\phi_{j+1}^{k+1/2} - \phi_j^{k+1/2}}{\Delta x} \right) - q_{j+1/2}^{k+1/2} \left( \frac{\mu_{j+1/2}^{k+1} + \mu_{j+1/2}^k}{2} \right) \\ & \quad \left( \frac{\phi_{j+1/2}^{k+1} - \phi_{j+1/2}^k}{\Delta y} \right) ] \\ & = \Delta y \sum_{j,k} (\mu_{j+3/2}^k (\zeta_{j+3/2} \phi_{j+3/2}^k)^{1/2} - \\ & - 2 \mu_{j+1/2}^k (\zeta_{j+1/2} \phi_{j+1/2}^k)^{1/2} + \mu_{j-1/2}^k (\zeta_{j-1/2} \phi_{j-1/2}^k)^{1/2})^2 \\ & + O(\Delta x + \Delta y + \Delta x \sup_k \sum_j |p_{j+1}^k - p_j^k|). \end{aligned}$$

The left side of (4.6) is an approximation of

$$\iint \left( \left( \frac{2}{3} p^3 + \frac{1}{2} q^2 \right) \phi_x - p q \phi_y \right) dx dy;$$



the right side is a nonnegative term plus a remainder which can be made small by suitable hypotheses, as in [13]. Thus we have shown

Theorem 4.1: Assume that as  $\Delta x, \Delta y \rightarrow 0$ , the discrete solution obtained from (4.3-4.5) remains uniformly bounded, and converges in measure to a limit pair of functions  $p, q$ , with  $p$  of bounded variation in  $x$ , uniformly with respect to  $y$ . Then for any nonnegative  $\phi \in C_0^\infty$ ,

$$(4.7) \quad \iint \left( \left( \frac{2}{3}p^3 + \frac{1}{2}q^2 \right) \phi_x - pq\phi_y \right) dx dy \geq 0,$$

which is the desired entropy inequality.

Next we consider the order of accuracy of the difference scheme. In regions where the solution is smooth and which are away from the sonic line ( $p=0$ ), each of the difference equations is clearly of second order accuracy. Thus we anticipate no trouble from the changing of the form of the difference equations near the  $x$ -boundaries.

In regions where the solution is not smooth, it still makes sense to determine to what order of accuracy a difference scheme approximates the weak form of a given differential equation. In the case of linear hyperbolic systems, this is sufficient to determine the order of magnitude of the error in distribution sense [13, 14]. To be precise, let

$q_h(x,y)$  be the function obtained by extension of the discrete variables  $q_{j+1/2}^{k+1/2}$ ;  $q_h$  is piecewise constant, and is our approximation to  $q$ . Similarly, let  $p_h^k$ ,  $k = 0,1,\dots,N$ , be a function of  $x$ , equal to the right side of (4.2), and let  $p_h = p_h(x,y)$  be the extension of the  $p_h^k$  with respect to  $y$ . The function  $p_h$  is piecewise linear in  $x$ , piecewise constant in  $y$ , and is our approximation to  $p$ .

Theorem 4.2: Suppose that as  $\Delta x, \Delta y \rightarrow 0$  with  $\Delta y/\Delta x$  fixed, the discrete solution  $p_h, q_h$  is uniformly bounded and that  $p_h$  is bounded variation in  $x$ , uniformly with respect to  $y$ . Then for any  $\phi \in C_0^\infty$ ,

$$(4.8) \quad \left| \iint (p_h^2 \phi_x - q_h \phi_y) dx dy \right| \leq O(\Delta x^2 + \Delta y^2),$$

and

$$(4.9) \quad \left| \iint (q_h \phi_x - p_h \phi_y) dx dy \right| \leq O(\Delta x^2 + \Delta y^2),$$

Proof: Let us begin with the second term in (4.8).

$$\begin{aligned} & - \iint q_h \phi_y dx dy = \iint q_{h,y} \phi dx dy \\ & = \sum_{j,k} (q_{j+1/2}^{k+1/2} - q_{j+1/2}^{k-1/2}) \int_{j\Delta x}^{(j+1)\Delta x} \phi(x, (k+\frac{1}{2})\Delta y) dx \\ & = \sum_{j,k} (q_{j+1/2}^{k+1/2} - q_{j+1/2}^{k-1/2}) \frac{1}{\Delta y} \int_{j\Delta x}^{(j+1)\Delta x} \int_{k\Delta y}^{(k+1)\Delta y} \phi(x,y) dx dy + O(\Delta x^2 + \Delta y^2) \end{aligned}$$

$$\begin{aligned}
 &= \Delta x \Delta y \sum_{j,k} \frac{(q_{j+1/2}^{k+1/2} - q_{j+1/2}^{k-1/2})}{\Delta y} \int_{k\Delta y}^{(k+1)\Delta y} \phi((j+\frac{1}{2})\Delta x, y) dy + O(\Delta x^2 + \Delta y^2) \\
 &\equiv \Delta x \Delta y \sum_{j,k} \frac{(q_{j+1/2}^{k+1/2} - q_{j+1/2}^{k-1/2})}{\Delta y} \alpha_{j+1/2}^k + O(\Delta x^2 + \Delta y^2),
 \end{aligned}$$

(4.10)

using the piecewise constant form of  $q_h$  and the smoothness of  $\phi$ . We note in passing that for the linear relation (4.9), both terms are treated in essentially this manner and (4.7) follows from the difference equation (4.4). In (4.8), however, the quadratic term is of central importance, and it is here that the peculiar interpretation (4.2) of the  $p_j^k$  is used. Let  $X$  be the space of continuous, piecewise linear functions of  $x$ , with nodes at the mesh points  $j\Delta x$ . For each  $k$ ,  $p_h^k \in X$ . Now the approximation of  $(p^2)_x$  in (4.5) is just what would be obtained (with respect to the  $x$ -variable) by Galerkin's method with the space  $X$ . (This is also an explanation of why the entropy inequality (4.6) was obtained.) Let  $\gamma^k \in X$  be the piecewise linear extension of the  $\alpha_{j+1/2}^k$ ; then

$$(4.11) \quad |\gamma^k(x) - \frac{1}{\Delta y} \int_{k\Delta y}^{(k+1)\Delta y} \phi(x, y) dy| = O(\Delta x^2 + \Delta y^2)$$

uniformly in  $x$ . Using the bounded variation of  $p_h$ , we have from (4.11)

$$\begin{aligned}
 \iint p_h^2 \phi_x dx dy &= -\Delta y \sum_k \left( p_h^k \right)_x^2 \frac{1}{\Delta y} \int_{k\Delta y}^{(k+1)\Delta y} \phi(x,y) dy dx \\
 (4.12) \quad &= -\Delta y \sum_k \left( p_h^k \right)_x^2 \gamma^k + O(\Delta x^2 + \Delta y^2) \\
 &= -\Delta x \Delta y \sum_{j,k} \frac{(\mu_{j+3/2}^k - \mu_{j-1/2}^k)(\mu_{j+1/2}^k + \mu_{j+1/2}^k + \mu_{j-1/2}^k)}{3\Delta x} \alpha_{j+1/2}^k + O(\Delta x^2 + \Delta y^2).
 \end{aligned}$$

Finally, multiply (4.5) by  $\Delta x \Delta y \alpha_{j+1/2}^k$  and sum over  $j, k$ . Using the smoothness of  $\phi$ , the artificial viscosity term gives a continuation of  $O(\Delta x^3 + \Delta y^3)$ , so (4.8) follows by comparison with (4.10) and (4.12). This concludes the proof.

Finally, we describe a potentially helpful modification of the difference scheme (4.3-4.5), corresponding to the simple change of the variable  $p \rightarrow p + \beta$  for a constant  $\beta$  to be determined empirically. The boundary conditions at  $x = 0$  and  $x = L$  become  $p = \beta - p_0$ , and an additional term

$$-2\beta \frac{p_{j+1}^k - p_j^k}{\Delta x}$$

is added to the left side of (4.5). This transformation is equivalent, in the difference scheme, to adding a dispersion term of order  $\beta \Delta x^2 p_{xxx}$ . While such a term will not aid in the generation of entropy in the vicinity of a shock, it may help in controlling the variation of  $p_h$ . This may be a serious problem in some applications; in particular, the fourth-order viscosity invariably leads to oscillatory, discrete shock profiles.

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Appendix

On the nonexistence of weak shock-type discontinuities in  
elliptic regions.

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Consider  $N$  conservation equations:

$$(A1) \quad \nabla_0 \underline{V}^i = A_x^i(x, y, \underline{u}) + B_y^i(x, y, \underline{u}) = 0 \quad i = 1, \dots, N$$

$$\text{for} \quad \underline{u} = \{u_j\} \quad j = 1, \dots, N$$

$$\underline{V}^i = (A^i, B^i)(x, y, \underline{u})$$

Let  $A^i, B^i$  be continuously differentiable ( $\in C^1$ ) in all their variables and:  $\frac{\partial A^i}{\partial u_j}$

$$A_j^i = \frac{\partial A^i}{\partial u_j}, \quad B_j^i = \frac{\partial B^i}{\partial u_j}$$

eq. (1) can be written as:

$$(A2) \quad A_j^i \frac{\partial u^j}{\partial x} + B_j^i \frac{\partial u^j}{\partial y} = 0$$

with summation convention on repeated indices.

Let (A1) be elliptic in a region  $E$  in  $\underline{u}$  space for  $(x, y)$  in  $\Omega_E$ , for which:

$$(A3) \quad \sum_{j=1}^N (\lambda A_j^i - \mu B_j^i) v^j = 0 \Rightarrow v^i \equiv 0$$

for  $\lambda, \mu$  real and not both zero.

Equation (A3) is a requirement that no real characteristics exist, or equivalently, that (A1) is elliptic. It is symmetric in  $(x,y)$  and  $N$  has to be even, i.e.  $N=2m$ .  
Let:

$$\underline{u}_1 \in E$$

and:

$S(\underline{u}_1)$  be the points on the shock polar emanating at  $\underline{u}_1$  (for which a shock jump from  $\underline{u}_1$  is compatible with the conservation laws (A1)).

Theorem (A1)

$\underline{u}_1$  is an isolated point of  $s(\underline{u}_1)$  (hence no 'weak' shocks near  $\underline{u}_1 \in E$  are possible).

Note that by continuity, there exists a neighborhood  $K$  of  $\underline{u}_1$  such that for all  $(\ell, m), (\underline{u}, \underline{v}) \in K \Rightarrow A^i, B^i(\underline{u})(\ell, m)$  will leave (A1) elliptic. Let  $K$  be convex, the claim is that there is no jump connection between  $\underline{u}_1$  and  $\underline{u}_k \in K$  for  $k \neq 1$ .

Proof:

Assume:  $\underline{u}_2 \in S(\underline{u}_1), \underline{u}_2 \in K, \sigma \neq 0$ .

Then:

$$\left. \frac{dy}{dx} \right|_{s.w} = \sigma = \frac{B^i(x, y, \underline{u}_1) - B^i(x, y, \underline{u}_2)}{A^i(x, y, \underline{u}_1) - A^i(x, y, \underline{u}_2)}$$

Let, for  $0 \leq \theta \leq 1$ :

$$A^i(\theta) = A^i(x, y, (1-\theta)\underline{u}_1 + \theta\underline{u}_2)$$

$$B^i(\theta) = B^i(x, y, (1-\theta)\underline{u}_1 + \theta\underline{u}_2)$$

we get:

$$(A^i, B^i)(0) = (A^i, B^i)(x, y, \underline{u}_1)$$

$$(A^i, B^i)(1) = (A^i, B^i)(x, y, \underline{u}_2)$$

and, using the mean-value theorem, there exist:

$\theta_A^i, \theta_B^i \quad i = 1, \dots, l$ , such that:

$$A^i(x, y, \underline{u}_2) - A^i(x, y, \underline{u}_1) = \frac{dA^i}{d\theta} \Big|_{\theta=\theta_A^i}$$

$$B^i(x, y, \underline{u}_2) - B^i(x, y, \underline{u}_1) = \frac{dB^i}{d\theta} \Big|_{\theta=\theta_B^i}$$

But we have:

$$\frac{dA^i}{d\theta} = \sum_{j=1}^N A_j^i(x, y, (1-\theta)\underline{u}_1 + \theta\underline{u}_2) (\underline{u}_2 - \underline{u}_1)$$

$$\frac{dB^i}{d\theta} = \sum_{j=1}^N B_j^i(x, y, (1-\theta)\underline{u}_1 + \theta\underline{u}_2) (\underline{u}_2 - \underline{u}_1)$$

and since:

$$(1-\theta)\underline{u}_1 + \theta\underline{u}_2 \in K$$

we have:

$$\sigma = \frac{\sum_{j=1}^N \lambda_j^i (\underline{u}_2^j - \underline{u}_1^j)}{\sum_{j=1}^N \beta_j^i (\underline{u}_2^j - \underline{u}_1^j)} \quad i = 1, \dots, N$$

for:

$$\lambda_j^i = A_j^i(x, y, (1-\theta_A)\underline{u}_1 + \theta_A \underline{u}_2)$$

$$\beta_j^i = B_j^i(x, y, (1-\theta_B)\underline{u}_1 + \theta_B \underline{u}_2)$$



letting:

$$\underline{v} = \underline{u}_2 - \underline{u}_1 \quad \in K$$

$$v^j = u_2^j - u_1^j \quad \in K$$

we get:

$$\sum_{j=1}^N \left( \lambda_j^i - \sigma \beta_j^i \right) v^j = 0$$

contradicting the ellipticity condition (A3).

This proves the non existence of arbitrarily weak shocks in elliptic fields, precludes the possibility of closed shock polars in  $\underline{u}$  space (e.g. the hodograph plane) and of "shocked" elliptic flows about obstacles (with shock strength vanishing at infinite).